THEOREMS ON THE PRODUCT OF TWO SUMMABILITY METHODS WITH APPLICATIONS —CORRIGENDUM

By C. T. RAJAGOPAL

Page **97**, lines 4,5 up: delete "and an integral".

Page **102**, line 4: for "$s(x) = 0$" read "$s(0) = 0$".

Page **102**, line 7: delete "monotonic decreasing".

Page **103**, foot-note last line: for "$\lambda_{n+1}\,\lambda_n \rightarrow 1$" read "$\lambda_{n+1}/\lambda_n \rightarrow 1$".

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ON MEASURABLE SETS

By T. P. SRINIVASAN

[Received January 28, 1954]

1. Introduction. Caratheodory's axiomatic approach to the theory of measure is well known. It consists in starting with an *outer measure* m^*, a set function defined axiomatically on the class of all subsets of a given set X, and singling out a particular class of subsets of X, called the *outer measurable* sets, in such a way that the restriction of m^* to this distinguished subclass is well-behaved. Alternatively, one can start with a well-behaved set function on a restricted class of sets,—for instance, the class of all intervals (assuming X to be the space of real numbers) and the set function on this class which assigns as value to each interval its length—construct an outer measure from this set function, and then define the distinguished class of outer measurable sets. The latter class will in general be very wide, and will in particular include the original class of sets [1, pp. 42-44]. A different but less well-known approach is the one considered by I. E. Segal [2]. Our object, in this paper, is to show that with suitable modifications, the class of measurable sets in the sense of Segal could be identified with the class of outer measurable sets in the sense of Caratheodory.

2. Statement of the theorem. We first state the definitions required to phrase our result.

We assume the basic notions of set theory. We shall call a set whose elements are themselves subsets of another set a *collection of sets*. A set, for us, subsequently will be a subset of a certain basic set X, given *a priori*.

A *ring* is a collection of sets closed under the formation of finite set unions and set differences.

This paper has its origin in a course of lectures on Modern Integration Theory given by Professor K. Chandrasekharan at the Tata Institute of Fundamental Research. My sincere thanks are due to him for his suggestions and criticism throughout the preparation of this paper

A *σ-ring* is a collection of sets closed under the formation of countable set unions and set differences.

A *σ-algebra* is a σ-ring, closed under the operation of taking complements in X.

A *measure* is a pair (R, r), where R is a ring of sets, and r a real-valued function on R, such that

(i) $0 \leqslant r \leqslant \infty$,

(ii) $r(\phi) = 0$, where ϕ denotes the void set, and

(iii) r is countably additive.

If (R, r) is a measure, a set E is said to be *r-null* if $E \in R$, and $r(E) = 0$. The measure (R, r) is said to be *complete*, if any subset of an r-null set is also r-null.

The measure (R, r) is said to be *finite*, if the set function r is strictly finite $(r < \infty)$.

A *measure space* is a finite measure (R, r) with the property that if $A_n \in R$, $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} r(A_n) < \infty$, then $\bigcup_{n=1}^{\infty} A_n \in R$.

The *completion* of a measure space (R, r) is a pair (R', r'), where R' is the collection of all sets E' which are of the form $E' = E \cup N$, where $E \in R$ and N is a subset of an r-null set, and r' is the set function on R' defined by $r'(E') = r(E)$, E' and E standing in the above relation. It is known [2] that the completion of a measure space is again a measure space, and that the underlying measure for this measure space is complete.

The *extension* of a measure space (R, r) is a pair $(\bar{R}, \bar{r})$, where $\bar{R}$ is the collection of all sets E which are such that the intersection of E with any set in R is again a set in R, and $\bar{r}$ is the set function on $\bar{R}$ defined by

$$\bar{r}(E) = \sup \{ r(A) / A \in R, A \subset E \}.$$

It is also known [2] that the extension $(\bar{R}, \bar{r})$ is a measure, where $\bar{R}$ is not merely a ring but a σ-algebra which includes R, and $\bar{r}$ agrees with r on R The sets in $\bar{R}$ will be called *r-measurable*. [These are the sets which are (extended) *measurable* in the sense of Segal,

for the measure space (R, r)]. If (R, r) is a measure space, we shall denote by $(\bar{R}', \bar{r}')$ the *extension* of the completion (R', r') of (R, r). The sets in $\bar{R}'$ will accordingly be called *r'-measurable.*

If (R, r) is a measure space, we define two set functions r^* and r_* on the collection of all sets as follows. For any set A, let K_A denote the collection of all sets $E \in R$ such that $E \supset A$, and let H_A denote the collection of all sets $E \in R$ such that $E \subset A$. Then

$$r^*(A) = \inf\{r(E)/E \in K_A\}, \text{ if } K_A \text{ is nonvoid,}$$

$$= \infty, \text{ otherwise;}$$

and $$r_*(A) = \sup\{r(E)/E \in H_A\}.$$

We may note the trivial properties of r^* and r_*, viz. that both are non-negative and monotone increasing, and both agree with r' on R', where (R', r') denotes the *completion* of (R, r), and in particular, with r on R.

We define a set E to be r^*-measurable, if for every set A, we have

$$r^*(A) = r^*(A \cap E) + r^*(A - E).$$

A set E is defined to be *r_*-measurable,* if for every set A for which $r_*(A) < \infty$, we have

$$r_*(A) = r_*(A \cap E) + r_*(A - E).$$

We shall denote by R^* the collection of all r^*-measurable sets, and by R_* the collection of all r_*-measurable sets.

THEOREM. *Let (R, r) be a measure space. Then, in the above notation, we have $R^* \equiv \bar{R}' \equiv R_*$. Further the restriction of r_* to R_* is identical with $\bar{r}'$, so that the pair $(\bar{R}', \bar{r}')$ is identical with (R_*, r_*).*

3. Preliminary lemmas. The proofs of the preliminary lemmas and of the main theorem will be simplified by the use of the concepts of *measurable cover* and *measurable kernel.* By a *measurable cover* of a set A, we mean a set $E \in R$ such that $E \supset A$ and $r(E) = r^*(A)$. By a *measurable kernel* of a set A, we mean a set $E \in R$ such that $E \subset A$ and $r(E) = r_*(A)$.

Lemma 1. *If (R, r) is a measure space, then R is closed for the formation of countable intersections of sets in it.*

This lemma is contained in [2] and we reproduce the proof only for the sake of completeness.

Let $A_n \in R, n=1, 2\ldots$, and $A = \bigcap_{i=1}^{\infty} A_i$. Let $B_i = \bigcap_{j \leqslant i} A_j$, $i = 1, 2\ldots$.

Then

$$A = \bigcap_{j=1}^{\infty} B_j = B_1 - \bigcup_{j=2}^{\infty} (B_1 - B_j).$$

But

$$\bigcup_{j=2}^{\infty} (B_1 - B_j) = \bigcup_{j=1}^{\infty} (B_j - B_{j+1}),$$

since $B_j \supset B_{j+1}$. Also

$$\sum_{j=1}^{\infty} r(B_j - B_{j+1}) = \sum_{j=1}^{\infty} \{r(B_j) - r(B_{j+1})\} \leqslant r(B_1) < \infty.$$

Hence, by the definition of a measure space, $\bigcup_{j=1}^{\infty} (B_j - B_{j+1})$ and consequently A belongs to R.

Lemma 2. *There exists a measurable cover for every set A for which $r^*(A) < \infty$, and there exists a measurable kernel for every set A for which $r_*(A) < \infty$.*

Let $r^*(A) < \infty$. Then by definition of r^*, there exists for each n, a set $E_n \in R$ such that $E_n \supset A$ and $r(E_n) < r^*(A) + 1/n$. If $E = \bigcap_{n=1}^{\infty} E_n$, then by Lemma 1, $E \in R$. Further $E \supset A$, and $r(E) \leqslant r(E_n) < r^*(A) + 1/n$ for every n. Hence $r(E) \leqslant r^*(A)$. The reverse inequality holds by the monotonicity of r^*. Hence $r(E) = r^*(E)$. Therefore E is a measurable cover of A.

The other part of the lemma is similarly proved.

Remarks. It may thus be noted that the existence of the measurable cover and measurable kernel is just a consequence of our particular method of definition of r^* and r_*, and it will be subsequently seen that it is only this fact that guarantees the desired result.

4. Proof of the main theorem. We may *first note*, besides the trivial properties of r^* and r_* already mentioned, that r^* is *subadditive* while r_* is *superadditive*, i.e. for any two sets A, B, we have $r^*(A \cup B) \leqslant r^*(A) + r^*(B)$, and for any two disjoint sets C, D, we have $r_*(C \cup D) \leqslant r_*(C) + r_*(D)$. For, if either of $r^*(A)$ and $r^*(B)$ is ∞, the relation $r^*(A \cup B) \leqslant r^*(A) + r^*(B)$ is trivially true, and in the alternative case, if E_A, E_B are measurable covers of A, B, which exist by Lemma 2, then $E_A \cup E_B \supset A \cup B$, and

$$r^*(A) + r^*(B) = r(E_A) + r(E_B) \geqslant r(E_A \cup E_B),$$

by the property of a measure,

$$= r^*(E_A \cup E_B) \quad \geqslant r^*(A \cup B).$$

Thus in any case r^* is subadditive. The superadditivity of r_* is similarly proved by considering measurable kernels.

We now show that $\bar{R}' \subset R^*$. Let $B \in \bar{R}'$. To show that $B \in R^*$ we need to verify that for every set A, we have $r^*(A) = r^*(A \cap B) + r^*(A - B)$. By virtue of the subadditivity of r^*, proved above, it only remains to verify that $r^*(A) \geqslant r^*(A \cap B) + r^*(A - B)$. This is trivially true if $r^*(A) = \infty$. If $r^*(A) < \infty$, let E be a measurable cover of A. Then $E \in R$ and hence $E \in R'$. Since $B \in \bar{R}'$, it follows that $B \cap E \in R'$. Since $\bar{R}'$ is a σ-algebra and $B \in \bar{R}'$, we have $X - B \in \bar{R}'$, and hence $E - B = (X - B) \cap E \in R'$. Further since $E \supset A$, we have $E \cap B \supset A - B$ and $E - B \supset A - B$. Thus

$$\begin{aligned} r^*(A) = r(E) = r'(E) &= r'(E \cap B) + r'(E - B) \\ &= r^*(E \cap B) + r^*(E - B) \\ &\geqslant r^*(A \cap B) + r^*(A - B). \end{aligned}$$

Hence it follows that $B \in R^*$, in other words $\bar{R}' \supset R^*$.

We now show that $R^* \subset \bar{R}'$. Let $B \in R^*$. To show that $B \in \bar{R}'$ we have only to show that $B \cap E \in R'$ for every set $E \in R'$. Now $r^*(B \cap E) \leqslant r^*(E) = r'(E) < \infty$, so that there exists a measurable cover F of $B \cap E$. We can assume that $F \subset E$, as otherwise $F \cap E$

is easily seen to serve the role of F. With this assumption we have $F \cap B = E \cap B$. Also, since $F \in R'$,

$$r'(F) = r^*(F) = r^*(F \cap B) + r^*(F - B), \qquad \text{since } B \in R^*,$$
$$\geqslant r^*(F \cap B) = r^*(E \cap B) = r(F) = r'(F).$$

Thus it follows that $r^*(F - B) = 0$. If now G is a measurable cover of $(F - B)$, $r(G) = 0$ and $G \supset F - B$. Hence $F - B$ is a subset of an r-null set and is therefore in R'. Thus $B \cap E = F \cap B = F - (F - B) \in R'$, and the proof is complete. Combining this with what we have shown above, we get $R^* \equiv \bar{R}'$.

The proof that $\bar{R}' \equiv R_*$ is almost a duplication of the above proof in principle, and only involves judicious choice of the measurable kernels.

We shall first show that $\bar{R}' \subset R_*$. Let $B \in \bar{R}'$. To show that $B \in R_*$, we have to verify that for every set A for which $r_*(A) < \infty$, we have $r_*(A) = r_*(A \cap B) + r_*(A - B)$. In this case there exists a measurable kernel of A and the proof proceeds on exactly the same lines as in the earlier case.

Let now $B \in R_*$. We shall prove that $B \in \bar{R}'$. For this, we need to show that $B \cap E \in R'$ for every set $E \in R'$. Now $r_*(E - B) < r_*(E) = r'(E) < \infty$, so that there exists a measurable kernel F of $E - B$. Let $F' = E - F$. Then

$$F' - B = (E - F) - B = (E - B) - F,$$

and

$$r_*(F' - B) = r_*((E - B) - F) \leqslant r_*(E - B) - r_*(F) = 0.$$

Further $r_*(F') = r_*(F' \cap B) + r_*(F' - B), \qquad \text{since } B \in R_*,$

$$= r_*(F' \cap B) = r_*(E \cap B).$$

If now F'' be a measurable kernel of $E \cap B$ then $F'' \subset E \cap B \subset F'$. Since $F', F'' \in R$ and $r(F') = r_*(E \cap B) = r(F'')$, it follows that $E \cap B \in R'$. This shows that $B \in \bar{R}'$ and thus $R_* \subset \bar{R}'$. Combining this with the previous result we conclude that $\bar{R}' \equiv R_*$. That the restriction of r_* to R_* is identical with r' is fairly obvious.

5. **Remarks.** 1. If (R, r) is any measure, the *outer measure* m^* induced by (R, r) in the sense of Caratheodory is defined thus. For *any* set A,

$$m^*(A) = \inf \{ \sum_{n=1}^{\infty} r(A_n)/A_n \in R, n = 1, 2\ldots, \text{and} \bigcup_{n=1}^{\infty} A_n \supset A \},$$

if such coverings of A exist in R,

$= \infty$, otherwise.

Let (R, r) be a *measure space*, and m^* the outer measure induced by (R, r). We wish to show that $m^* \equiv r^*$ in our notation. That $m^* \leqslant r^*$ is immediate from the definitions of m^*, r^*. The opposite inequality $m^*(E) \geqslant r^*(E)$ is trivially true for sets E for which $m^*(E) = \infty$. Let now $m^*(E) < \infty$. Then, by the definition of m^*,

$$m^*(E) = \inf \{ \sum_{n=1}^{\infty} r(A_n)/A_n \in R, n = 1, 2\ldots, \bigcup_{n=1}^{\infty} A_n \supset E \},$$

and further this infimum need be taken only over the class of such sequences $\{A_n\}$ as have the further property that $\sum_{n=1}^{\infty} r(A_n) < \infty$. In this case $A = \bigcup_{n=1}^{\infty} A_n \in R$, and $r(A) \leqslant \sum_{n=1}^{\infty} r(A_n)$, since (R, r) has been assumed to be a measure space. It is now immediate that

$$r^*(E) = \inf \{ r(A)/A \in R, A \supset E \}$$
$$\leqslant \inf \{ \sum_{n=1}^{\infty} r(A_n)/A_n \in R, n=1, 2, \ldots \bigcup_{n=1}^{\infty} A_n \supset E \} = m^*(E),$$

and the proof that $r^* \equiv m^*$ is complete.

It is now immediate that the collection R^* of r^*-measurable sets is identical with the collection M^* of outer measurable sets, since outer measurability for a set is defined in the same way as r^*-measurability with r^* replaced by m^*. It is well known that, for any outer measure m^*, the pair (M^*, m^*) is a complete measure and hence the pair (R^*, r^*) is also so. The extension $(\bar{R}, \bar{r})$ of the measure space (R, r), on the other hand, is not in general a complete measure. We have now shown that this is precisely the crux of the difference between $(\bar{R}, r)$ and (R^*, r^*), in other words between the

approaches of Segal and Caratheodory, and that by *completing* $(\bar{R}, \bar{r})$ in a particular way to the measure $(\bar{R}', \bar{r}')$, the two classes $\bar{R}'$ and R^* could be identified. It should be noted however that the set functions r^* and $\bar{r}'$ are not identical on $\bar{R}'$, while r_* is identical with $\bar{r}'$ on $\bar{R}'$. Our next remark shows by how much r^* differs from $\bar{r}'$ on this class.

2. If $A \in \bar{R}'$, and is contained in a countable union of sets in R', say $A \subset \bigcup_{n=1}^{\infty} E_n$, $E_n \in R$, $n = 1, 2, \ldots$, then $A = \bigcup_{n=1}^{\infty} (A \cap E_n)$ and each $A \cap E_n \in R'$ since $A \in \bar{R}'$. Since any countable union of sets in a ring could be expressed as a countable disjoint union of sets in it, it follows that $A = \bigcap_{n=1}^{\infty} B_n$, $B_n \in R'$, $B_m \cap B_n = \phi$, $m \neq n$. Hence $r^*(A) = \sum_{n=1}^{\infty} r^*(B_n) = \sum_{n=1}^{\infty} r'(B_n) = \bar{r}'(A)$. Thus on such sets A, $r^*(A) = \bar{r}'(A)$. In particular, if X is contained in a countable union of sets in R, then the restriction of r^* to $\bar{R}'$ is identical with $\bar{r}'$. In other cases, a pathological situation may arise, viz. there may exist sets $A \in \bar{R}'$ such that $r^*(A) = \infty$ and $\bar{r}'(A) = 0$. In fact any r^*-null set which is not in R' is so, and it is precisely the existence of such sets that causes any distinction between the measures $(\bar{R}', \bar{r}')$, and $(\bar{R}', \bar{r}^*)$.

3. Since $(\bar{R}', \bar{r}')$ is a measure, and (R_*, r_*) has been shown to be identical with $(\bar{R}', \bar{r}')$, it follows that (R_*, r_*) is a measure. The direct proof of this fact and of the relation $R_* \equiv R^*$, without introducing the intermediary pair $(\bar{R}', \bar{r}')$, is less simple than our present method.

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ON THE SUMMABILITY OF EIGENFUNCTION EXPANSIONS I

By T. V. AVADHANI

[Received January 13, 1954]

1. Introduction. In the Euclidean space E_k of k dimensions ($k \geqslant 2$) let D denote a bounded open domain whose boundary B is assumed to be sufficiently regular, so that the following boundary value problem

$$\sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} + \lambda u = 0, \qquad \text{in } D$$

$$u(x) \equiv u(x_1, x_2, \ldots x_k) = 0, \qquad \text{on } B$$

possesses an infinite number of positive eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \ldots \leqslant \lambda_n \to \infty$ with the corresponding eigenfunctions $w_1(x), w_2(x), \ldots, w_n(x), \ldots$. We assume that the eigenfunctions $\{w_n(x)\}$ form a complete orthonormal set in the space (L^2) of functions $f(x)$ which are square integrable in D. Now given an arbitrary function $f(x) \in L^2$ and defined in D we form its Fourier eigenfunction series

$$f(x) \sim \Sigma a_n w_n(x), \tag{1.1}$$

where

$$a_n = \int_D f(x)\, w_n(x)\, dx, \tag{1.2}$$

and dx denotes the element of volume in E_k.

If $f(x) \in L^2$ Riesz summability (λ, a) of the series (1.1) at the point x_0 in any bounded domain D was proved to depend only on the behaviour of $f(x)$ in the neighbourhood of x_0 if $a > k/2$ [5] Recently Titchmarsh [6] pointed out that a could be equal to $k/2$. This paper establishes that it is enough if $a \geqslant (k-1)/2$. In the case of multiple Fourier series—that is in the case where the domain D is a square—the same property was proved by Bochner [2].

This paper owes in fact its origin to the paper of Titchmarsh [6]. His modified argument is adopted in this paper together with a recent result of Avakumović [1] on the asymptotic expansion of eigenfunctions.

2. Preliminary lemmas. The following properties of Bessel functions [7] are used in this paper.

$$J_\mu(x) = O(x^{-1/2}), x \to \infty. \tag{2.1}$$

For $\nu > -1$, $\delta > -1$, and $a < b$

$$a^{1/2} b^{1/2} t J_\nu(a t) J_\delta(b t) \sim \psi(t) + O(a^{-1} t^{-1}), \tag{2.2}$$

for large values of the arguments where

$$\psi(t) = A_0 \cos (a+b) t + B_0 \cos (a-b) t + + C_0 \sin (a+b) t + D_0 \sin (a-b) t, \tag{2.3}$$

and A_0, B_0, C_0, D_0, are constants.

If a and b are real and positive $\nu > -1$, and $\mu - \nu > 0$

$$\int_0^\infty J_{\mu+1}(a t) J_\nu(b t) t^{\nu-\mu} dt = \begin{cases} 0, \text{ for } a < b \\ \dfrac{b^\nu (a^2 - b^2)^{\mu-\nu}}{2^{\mu-\nu} a^{\mu+1} \Gamma(\mu - \nu + 1)}, \text{ for } a \geqslant b. \end{cases} \tag{2.4}$$

We shall also use the following notation and formulae in the sequel.

The spherical average of the function $f(x)$ over a sphere of radius t and centre x is

$$f(x; t) = \frac{1}{\sigma_{k-1}} \int_S f(x + \epsilon t) \, d\sigma, \tag{2.5}$$

where $x + \epsilon t = (x_1 + \epsilon_1 t, x_2 + \epsilon_2 t, \ldots, x_k + \epsilon_k t)$, S is the sphere $\sum_{i=1}^{k} \epsilon_i^2 = 1$, $d\sigma$ the $k-1$ dimensional volume element of S and σ_{k-1} the volume of S [3].

$$F(R) \equiv \sum_{\lambda_n \leqslant R} w_n^2(x) = D_k R^{k/2} + O(R^{(k-1)/2}), \tag{2.6}$$

where D_k is a number depending on the volume of the domain D [1]. The following result can easily be deduced from (2.6). If $p \geqslant 0$

$$\sum_{R-\sqrt{R}<\lambda_n\leqslant R+\sqrt{R}} \frac{w_n^2}{\lambda_n^p} = O(R^{k/2-p-1/2}). \tag{2.7}$$

If σ is the surface of the sphere with centre x and radius t, $|\sigma|$ is the $k-1$ dimensional volume of the surface, then [4]

$$\frac{1}{|\sigma|}\int_\sigma w_n\, d\sigma = \frac{w_n(x)\,\Gamma(k/2)\,J_{k/2-1}(t\sqrt{\lambda_n})}{(\frac{1}{2}\,t\sqrt{\lambda_n})^{k/2-1}}. \tag{2.8}$$

Lemma 1. *If* $\mu > -1, \nu > -1, p > 0$ *and* $\delta > 0$,

$$\int_\delta^\infty t^{-p}\, J_\mu(a\,t)\, J_\nu(b\,t)\, dt = O(a^{-1/2}\, b^{-1/2}),$$

for all large values of a and b.

Since $p > 0$ the lemma follows by using (2.1). A more precise result can be obtained if $a \neq b$.

Lemma 2. *If* $\mu > -1, \nu > -1, p > -1, \delta > 0$ *and* $a \neq b$

$$\int_\delta^\infty t^{-p}\, J_\mu(a\,t)\, J_\nu(b\,t)\, dt = O\left(\frac{a^{-1/2}\, b^{-1/2}}{|b-a|}\right), \tag{2.9}$$

if $|b-a| \to \infty$.

Proof. Expressing the product of the two Bessel functions in the integrand on the left side of (2.9) in the form of (2.2) and using (2.3) we observe that in the asymptotic expansion of the left side of (2.9) the leading terms will contain integrals of the type

$$\text{constant}\,(a^{-1/2}\,b^{-1/2})\left[\int_\delta^\infty t^{-p-1} \begin{matrix}\sin\\ \cos\end{matrix}\,(a-b)\,t\,dt\right] = O\left(\frac{a^{-1/2}\,b^{-1/2}}{|b-a|}\right)$$

on integration by parts.

Lemma 3. (i) *If* $\psi(t)$ *is a continuous function of* t *in* (a, b) *and* $F(t)$ *a step-function defined as*

$$F(t) = \begin{cases} 0, & \text{if } 0 \leqslant t < \lambda_1, \\ \sum_{i=1}^{n} w_i^2, & \text{if } \lambda_n \leqslant t < \lambda_{n+1}, \end{cases}$$

then

$$\int_a^b \psi(t)\, d\,F(t) = \sum_{a<\lambda_n\leqslant b} w_n^2\, \psi(\lambda_n). \tag{2.10}$$

(ii) *If the limit as* $b \to \infty$ *of the integral on the left side of* (2.10) *exists,* $\psi(t)$ *being continuous in* (a, ∞), *then*

$$\int_a^\infty \psi(t)\; dF(t) = \sum_{\lambda_n>a} w_n^2\, \psi(\lambda_n). \tag{2.11}$$

Refer [8, Ch. I, §§ 6, 9].

Lemma 4. *For a fixed* $N > \lambda_1$ *and* $R \to \infty$

$$\sum_{N<\lambda_n\leqslant R-\sqrt{R}} \frac{w_n^2}{\lambda_n^{(k-1)/2}\,(\sqrt{R}-\sqrt{\lambda_n})^2} + \sum_{\lambda_n>R+\sqrt{R}} \frac{w_n^2}{\lambda_n^{(k-1)/2}\,(\sqrt{\lambda_n}-\sqrt{R})^2} = O(1), \tag{2.12}$$

where the λ_n*'s are the eigenvalues and the* w_n*'s are the corresponding eigenfunctions.*

Proof. Choosing

$$\psi(t) = \frac{1}{t^{(k-1)/2}\,(\sqrt{t}-\sqrt{R})^2},$$

which is continuous both in $(N, R-\sqrt{R})$ and $(R+\sqrt{R}, \infty)$ and applying Lemma 3, we observe from (2.10) and (2.11) that the left side of (2.12) is

$$\int_N^{R-\sqrt{R}} \frac{dF(t)}{t^{(k-1)/2}(\sqrt{R}-\sqrt{t})^2} + \int_{R+\sqrt{R}}^\infty \frac{dF(t)}{t^{(k-1)/2}(\sqrt{t}-\sqrt{R})^2}. \tag{2.13}$$

Integrating by parts and rearranging the terms we obtain that (2.13) is

$$\left.\frac{F(t)}{t^{(k-1)/2}\,(\sqrt{R}-\sqrt{t})^2}\right]_N^{R-\sqrt{R}} + \left.\frac{F(t)}{t^{(k-1)/2}\,(\sqrt{t}-\sqrt{R})^2}\right]_{R+\sqrt{R}}^\infty -$$
$$- \int_N^{R-\sqrt{R}} \frac{F(t)\,dt}{t^{k/2}\,(\sqrt{R}-\sqrt{t})^3} + \int_{R+\sqrt{R}}^\infty \frac{F(t)\,dt}{t^{k/2}\,(\sqrt{t}-\sqrt{R})^3} +$$
$$+ \frac{k-1}{2}\left\{\int_N^{R-\sqrt{R}} \frac{F(t)\,dt}{t^{(k+1)/2}\,(\sqrt{R}-\sqrt{t})^2} + \right.$$
$$\left. + \int_{R+\sqrt{R}}^\infty \frac{F(t)\,dt}{t^{(k+1)/2}\,(\sqrt{t}-\sqrt{R})^2}\right\}. \tag{2.14}$$

Substituting the asymptotic expansion of $F(t)$ from (2.6) we observe that the first two terms of (2.14) are

$$- D_k \left\{ \frac{(R+\sqrt{R})^{1/2}}{(\sqrt{R+\sqrt{R}}-\sqrt{R})^2} - \frac{(R-\sqrt{R})^{1/2}}{(\sqrt{R-\sqrt{R}}-\sqrt{R})^2} \right\} + O(1) = O(1). \tag{2.15}$$

Integrating by parts each one of the remaining integrals it can be shown that they are

$$o(1). \tag{2.16}$$

From (2.15) and (2.16) the lemma follows easily.

3. Summability theorems. Take Parseval's formula

$$\int_D f(x)\, g(x)\, dx = \Sigma\, a_n\, b_n, \tag{3.1}$$

where

$$f \sim \Sigma\, a_n\, w_n, \qquad f \in L^2,$$
$$g \sim \Sigma\, b_n\, w_n, \qquad g \in L^2.$$

Choose

$$g(x) = \begin{cases} 0, \text{ if } |x - x_0|^2 = t^2 > \delta^2, \\ \dfrac{2^a \Gamma(a+1)\, R^{k/2}}{(2\pi)^{k/2}}\, V_{a+k/2}(t\sqrt{R}), \text{ if } |x-x_0|^2 = t^2 \leqslant \delta^2 \end{cases}$$

$$\text{and } a > -1/2, \tag{3.2}$$

where $|x - y|^2 = (x_1 - y_1)^2 + \dots + (x_k - y_k)^2$, $0 < \delta < \delta'$, δ' being the minimum distance of x_0 from the boundary B, and

$$V_{a+k/2}(t\sqrt{R}) = (t\sqrt{R})^{-(a+k/2)} J_{a+k/2}(t\sqrt{R}).$$

Combining (3.2) and (2.5) we have

$$\int_D f(x)\, g(x)\, dx = \frac{2^{a-k/2+1}\, \Gamma(a+1)\, R^{k/2}}{\Gamma(k/2)} \times$$
$$\times \int_0^\delta t^{k-1}\, V_{a+k/2}(t\sqrt{R})\, f(x_0;\, t)\, dt. \tag{3.3}$$

Employing (2.8) we easily obtain that

$$b_n = \frac{2^a\,\Gamma(a+1)\,w_n(x_0)}{R^{a/2-k/4}\,\lambda_n^{k/4-1/2}} \int_0^\delta t^{-a} J_{a+k/2}(t\sqrt{R})\,J_{k/2-1}(t\sqrt{\lambda_n})\,dt$$

$$= \frac{2^a\,\Gamma(a+1)\,w_n(x_0)}{R^{a/2-k/4}\,\lambda_n^{k/4-1/2}} \left\{ \int_0^\infty - \int_\delta^\infty \right\}$$

$$= \left(1-\frac{\lambda_n}{R}\right)^a w_n(x_0) - \frac{2^a\,\Gamma(a+1)\,w_n(x_0)}{R^{a/2-k/4}\,\lambda_n^{k/4-1/2}} \times$$

$$\times \int_\delta^\infty t^{-a} J_{a+k/2}(t\sqrt{R})\,J_{k/2-1}(t\sqrt{\lambda_n})\,dt, \quad (3.4)$$

in view of (2.4) for $a > 0$. The first term in (3.4) will be zero if $\lambda_n > R$. Combining (3.1), (3.3) and (3.4) we have for $a > 0$

$$\sum_{\lambda_n \leqslant R} \left(1-\frac{\lambda_n}{R}\right)^a a_n\,w_n(x_0) - \frac{2^{a-k/2+1}\,\Gamma(a+1)\,R^{k/2}}{\Gamma(k/2)} \times$$

$$\times \int_0^\delta t^{k-1}\,V_{a+k/2}(t\sqrt{R})\,f(x_0;t)\,dt$$

$$= \frac{2^a\,\Gamma(a+1)}{R^{a/2-k/4}} \sum \frac{a_n\,w_n(x_0)}{\lambda_n^{k/4-1/2}} \times$$

$$\times \int_\delta^\infty t^{-a} J_{a+k/2}(t\sqrt{R})\,J_{k/2-1}(t\sqrt{\lambda_n})\,dt. \quad (3.5)$$

Let $R > 2N$, where N is chosen such that given an $\epsilon > 0$

$$\sum_{\lambda_n > N} a_n^2 < \epsilon.$$

The right side of (3.5) can be written as

$$\frac{2^a\,\Gamma(a+1)}{R^{a/2-k/4}} \left\{ \sum_{\lambda_n \leqslant N} + \sum_{\lambda_n > N} \right\}. \quad (3.6)$$

The first part of (3.6) contains only a finite number of terms a typical term being

$$\frac{a_n\,w_n(x_0)}{\lambda_n^{k/4-1/2}} \int_\delta^\infty t^{-a} J_{a+k/2}(t\sqrt{R})\,J_{k/2-1}(t\sqrt{\lambda_n})\,dt = O(R^{-3/4}),$$

as $a > -1/2$. Therefore

$$\left| \sum_{\lambda_n \leqslant N} \right| = O(R^{-3/4}). \quad (3.7)$$

Applying Schwartz's inequality to the second part of (3.6) we observe that

$$\Big|\sum_{\lambda_n>N}\Big| \leqslant \Big(\sum_{\lambda_n>N} a_n^2\Big)^{1/2}\Big\{\sum_{\lambda_n>N}\frac{w_n^2(x)}{\lambda_n^{k/2-1}}\times$$

$$\times\Big(\int_\delta^\infty t^{-a}J_{a+k/2}(t\sqrt{R})\,J_{k/2-1}(t\sqrt{\lambda_n})\,dt\Big)^2\Big\}^{1/2}$$

$$= O(\epsilon)^{\frac{1}{2}}\Big\{\sum_{N<\lambda_n\leqslant R-\sqrt{R}}\!\!{}_1 + \sum_{R-\sqrt{R}<\lambda_n\leqslant R+\sqrt{R}}\!\!{}_2 + \sum_{\lambda_n>R+\sqrt{R}}\!\!{}_3\Big\}^{1/2}. \qquad (3.8)$$

Applying (2.9) for the integrals in Σ_1 and Σ_3 of (3.8) and then employing Lemma 4 we observe that

$$\Sigma_1 + \Sigma_3 = O(R^{-1/2}). \qquad (3.9)$$

Using Lemma 1 for the integral in Σ_2 of (3.8), and then employing (2.7) we observe that

$$\Sigma_2 = O(R^{-1/2}). \qquad (3.10)$$

Combining (3.8), (3.9) and (3.10) we observe that

$$\Big|\sum_{\lambda_n>N}\Big| = O(\epsilon^{1/2}R^{-1/4}). \qquad (3.11)$$

From (3.7) and (3.11) it follows that (3.6) is

$$O(R^{-a/2+k/4-3/4}) + O(\epsilon^{1/2}R^{-a/2+k/4-1/4}) = o(1), \qquad \text{if } a \geqslant (k-1)/2. \qquad (3.12)$$

We therefore have the following theorems.

THEOREM 1. *If*

$$\Sigma a_n^2 < \infty,$$

then to every point x_0 in D there corresponds a positive number δ' such that for $0<\delta<\delta'$ and $R\to\infty$

$$\sum_{\lambda_n\leqslant R}\Big(1-\frac{\lambda_n}{R}\Big)^a a_n\,w_n(x_0) = \frac{2^{a-k/2+1}\,\Gamma(a+1)\,R^{k/2}}{\Gamma(k/2)}\times$$

$$\times\int_0^\delta t^{k-1}\,V_{a+k/2}(t\sqrt{R})\,f(x_0;t)\,dt + o(1), \qquad (3.13)$$

for $a\geqslant(k-1)/2$, where $V_\mu(x) = x^{-\mu}J_\mu(x)$ and $f(x_0;t)$ is the spherical average of the function $f(x)$ on the sphere of radius t and centre x_0.

THEOREM 2. *If $f(x) \in L^2$, the Riesz summability (λ, a) of the eigen-function series $\Sigma a_n w_n(x)$ depends only on the behaviour of $f(x)$ in the neighbourhood of x_0, if $a \geqslant (k-1)/2$.*

Writing the right side of (3.13) as

$$\frac{2^{a-k/2+1}\,\Gamma(a+1)\,R^{k/2}}{\Gamma(k/2)}\int_0^{\delta} t^{k-1}\,V_{a+k/2}(t\sqrt{R})\,l\,dt+$$

$$+\frac{2^{a-k/2+1}\,\Gamma(a+1)\,R^{k/2}}{\Gamma(k/2)}\int_0^{\delta} t^{k-1}\,V_{a+k/2}(t\sqrt{R})\,\{f(x_0;t)-l\}\,dt, \quad (3.14)$$

we observe that the first term of (3.14) tends to l as $R \to \infty$. Splitting the second term as

$$\frac{2^{a-k/2+1}\,\Gamma(a+1)\,R^{k/2}}{\Gamma(k/2)}\int_0^{1/\sqrt{R}} t^{k-1}\,V_{a+k/2}(t\sqrt{R})\,\{f(x_0;t)-l\}\,dt+$$

$$+\frac{2^{a-k/2+1}\,\Gamma(a+1)\,R^{k/2}}{\Gamma(k/2)}\int_{1/\sqrt{R}}^{\delta} t^{k-1}\,V_{a+k/2}(t\sqrt{R})\,\{f(x_0;t)-l\}\,dt$$

$$=I_1+I_2, \quad (3.15)$$

we observe that

$$I_2=O\left(R^{k/2}\int_{1/\sqrt{R}}^{\delta} t^{k-1}(t\sqrt{R})^{-a-k/2-1/2}\,|f(x_0;t)-l|\,dt\right)$$

$$=O\left(R^{-a/2+k/4-1/4}\int_{1/\sqrt{R}}^{\delta}\frac{d\,P}{t^{a+k/2+1/2}}\right), \quad (3.16)$$

where

$$P(t)=\int_0^t s^{k-1}\,|f(x_0;s)-l|\,ds.$$

Integrating by parts the integral on the right side of (3.16) we obtain that

$$I_2=O(R^{-a/2+k/4-1/4})\left[\left(\frac{P(t)}{t^{a+(k+1)/2}}\right)_{1/\sqrt{R}}^{\delta}+\int_{1/\sqrt{R}}^{\delta}\frac{P(t)\,dt}{t^{a+(k+3)/2}}\right]$$

$$=o(1), \quad (3.17)$$

as $R \to \infty$ if $a>(k-1)/2$. In I_1 since $|V_{a+k/2}(t\sqrt{R})|=O(1)$ we observe that $I_1 \to 0$ as $R \to \infty$ if

$$\lim_{R\to\infty} R^{k/2}\int_0^{1/\sqrt{R}} t^{k-1}\,|f(x_0;t)-l|\,dt=0, \quad (3.18)$$

Combining (3.13), (3.14), (3.15), (3.17), and (3.18) we have the following theorem.

THEOREM 3. *If $f \in L^2$ and if for a fixed point x_0 in D*

$$\lim_{\eta \to 0} \frac{1}{\eta^k} \int_0^\eta t^{k-1} |f(x_0; t) - l| \, dt = 0,$$

then

$$\lim_{R \to \infty} \sum_{\lambda_n \leqslant R} \left(1 - \frac{\lambda_n}{R}\right)^\alpha a_n \, w_n \, (x_0) = l, \textit{ for } \alpha > \frac{k-1}{2}.$$

However substituting the asymptotic expansion of $V_{\alpha+k/2}$ $(t \sqrt{R})$ in I_2 of (3.15) we have

$$I_2 = \frac{2^{\alpha - k/2+1}\, \Gamma(\alpha+1)\, R^{k/2}}{\Gamma(k/2)} \int_{1/\sqrt{R}}^{\delta} t^{k-1} \, (t \sqrt{R})^{-\alpha-k/2-1/2} \times$$
$$\times \cos\, (t \sqrt{R}) \, \{f(x_0; t) - l\} \, dt +$$
$$+ \, O\left(R^{k/2} \int_{1/\sqrt{R}}^{\delta} t^{k-1} \, (t \sqrt{R})^{-\alpha-k/2-3/2} |f(x_0; t) - l| \, dt\right). \quad (3.19)$$

Integrating by parts and proceeding as in (3.16) to (3.17) it is easily seen that the second integral on the right side of (3.19) is $o(1)$ as $R \to \infty$ if $\alpha > (k-3)/2$. The first integral on the right side of (3.19) tends to zero as $R \to \infty$ if the usual Lebesgue's condition is satisfied, i.e. given an $\epsilon > 0$ if

$$\lim_{\eta' \to 0} \int_{\eta'}^{\epsilon} \frac{|f(x_0; t) - f(x_0; t + \eta')|}{t} \, dt = 0.$$

We therefore have

THEOREM 4. *If $f \in L^2$ and if for a fixed point x_0 in D*

$$\lim_{\eta \to 0} \frac{1}{\eta^k} \int_0^\eta t^{k-1} |f(x_0; t) - l| \, dt = 0,$$

and if for some fixed $\epsilon > 0$

$$\lim_{\eta' \to 0} \int_{\eta'}^{\epsilon} \frac{|f(x_0; t) - f(x_0; t + \eta')|}{t} \, dt = 0,$$

then

$$\lim_{R\to\infty} \sum_{\lambda_n \leqslant R} \left(1 - \frac{\lambda_n}{R}\right)^a a_n w_n(x_0) = l, \; for \; a = \frac{k-1}{2} .$$

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ON EXCEPTIONAL VALUES OF ENTIRE FUNCTIONS OF INFINITE ORDER

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We define the k-th order of an entire function $f(z)$ by

$$\rho_k = \overline{\lim_{r\to\infty}} \frac{l_{k+1} M(r)}{\log r},$$

where $M(r)$ has the usual meaning; and $l_2\, x = \log\log x$, $l_3\, x = \log\log\log x$ and so on.

THEOREM. *If $f(z)$ be an entire function of infinite k-th order, but of finite $(k+1)$-th order, there exists, at most, one entire function $f_1(z)$ of finite k-th order (including a constant), such that the product of primary factors formed with the zeros of the function $f(z) - f_1(z)$ is of finite k-th order.*

PROOF. Let us suppose that there exist two entire functions $f_1(z)$ and $f_2(z)$ of finite k-th order, which satisfy this condition. Then

$$f(z) - f_1(z) = z^{k_1} P_1(z)\, e^{Q_1(z)}, \tag{1}$$

and

$$f(z) - f_2(z) = z^{k_2} P_2(z)\, e^{Q_2(z)}, \tag{2}$$

where $P_1(z)$ and $P_2(z)$ are products of primary factors, of finite k-th order, and $Q_1(z)$ and $Q_2(z)$ are entire functions

Subtracting, we have

$$f_2(z) - f_1(z) = z^{k_1} P_1 e^{Q_1} - z^{k_2} P_2 e^{Q_2}. \tag{3}$$

Denoting the left-hand side by f_0, we have

$$R_1 e^{Q_1} - R_2 e^{Q_2} = f_0, \tag{4}$$

where R_1 and R_2 stand for $z^{k_1} P_1$ and $z^{k_2} P_2$ respectively.

Taking logarithms and differentiating (4), we have

$$f_0 \{(R'_1 + R_1 Q'_1) e^{Q_1} - (R'_2 + R_2 Q'_2) e^{Q_2}\} = f'_0 (R_1 e^{Q_1} - R_2 e^{Q_2}). \quad (5)$$

Multiplying throughout by e^{-Q_1}, we get

$$\frac{f_0 (R'_1 + R_1 Q'_1) - f'_0 R_1}{f_0 (R'_2 + R_2 Q'_2) - f'_0 R_2} = e^{Q_2 - Q_1} = \frac{R_1 - f_0 e^{-Q_1}}{R_2}. \quad (6)$$

We can easily prove that each of the functions $Q_1(z)$ and $Q_2(z)$ is of infinite $(k-1)$-th order. For if either of them, say, $Q_1(z)$ be of finite $(k-1)$-th order, by (1), we have

$$f(z) = f_1(z) + R_1(z) e^{Q_1(z)}.$$

Since each of the functions, $f_1(z)$, $R_1(z)$ and $e^{Q_1(z)}$ is of finite k-th order, it follows that $f(z)$ is of finite k-th order, which is contrary to the hypothesis.

Also, we have

$$e^{Q_1(z)} = \frac{f(z) - f_1(z)}{R_1(z)}.$$

Since each of the functions on the right-hand side is of finite $(k+1)$-th order, therefore, if $A(r)$ denotes the maximum value of Re $Q_1(z)$ on the circle $|z| = r$, we have

$$\varlimsup_{r\to\infty} \frac{l_{k+1} A(r)}{\log r} = B,$$

where B is a finite number. Now, by a theorem of Borel and Carathéodory [1, § 5.5], we can easily prove that $Q_1(z)$ is of finite k-th order. Similarly, we can prove that $Q_2(z)$ is of finite k-th order. Consequently, we can show that each of the functions $Q'_1(z)$ and $Q'_2(z)$ is of finite k-th order.

Therefore, by (6), it follows that each of the functions $e^{Q_2 - Q_1}$ and e^{-Q_1} is of finite k-th order. [If f and g are two entire functions of finite k-th order, and f/g is also entire, then f/g is of finite k-th order. This follows by using Nevanlinna's characteristic function $T(r, f)$ and noting that $T(r, f/g) = T(r, f) + T(r, 1/g) + O(1) = T(r, f) + T(r, g) + O(1)$, and that when f is an entire function, $T(r, f) \leqslant \log M(r, f) \leqslant 3\, T(2r, f)$].

Consequently, by the theorem (loc. cit.), we can prove that each of the functions $Q_1(z)$ and $Q_2(z)$ is of finite $(k-1)$-th order. This is a contradiction, and so the theorem is proved.

COROLLARY. *If $f(z)$ is an entire function of infinite order, there exists, at most, one entire function $f_1(z)$ of finite order (including a constant), such that the exponent of convergence of the zeros of the function $f(z) - f_1(z)$ is finite.*

REMARK. When (5) is multiplied by e^{-Q_1}, the co-efficient of $e^{Q_2 - Q_1}$ cannot vanish identically. For, if it vanish, we have

$$f_0(R'_2 + R_2 Q'_2) - f'_0 R_2 = 0,$$

or

$$\frac{f'_0 R_2 - f_0 R'_2}{R_2^2} \frac{R_2}{f_0} = Q'_2.$$

Solving this differential equation, we get

$$f_0 = H.\ R_2\, e^{Q_2}$$

or

$$f(z) = A\, f_1(z) + D\, f_2(z),$$

where H, A and D are constants.

But this is impossible, since $f(z)$ is of infinite k-th order, and each of the functions $f_1(z)$ and $f_2(z)$ is of finite k-th order.

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OPEN TOPOLOGICAL DISKS IN THE PLANE

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1. The almost fixed point property. A subset X of a metric space (M, d) has the "almost fixed point property" (abbreviated AFPP if and only if for each continuous function f of X into X and each positive number ϵ, there exists a point $p \in X$ for which $d\ (p, f(p))$ is less than ϵ.

It is easily seen that for compact sets the AFPP is equivalent to the ordinary fixed point property.

The AFPP is not a topological property. That is, it is possible for a set X which has the AFPP to be homeomorphic to a set Y which does not have the AFPP.

We are interested in determining whether or not certain non-compact sets have the AFPP. The principal result obtained in this paper is that a bounded open topological disk in the plane has the AFPP if the boundary of the disk is locally connected.

2. Some theorems about open topological disks. We let D be the set of all points z in the complex plane for which $|z| < 1$, and let D^* be the closure of D. A subset of the plane which is homeomorphic to D is called an *open topological disk*; a subset which is homeomorphic to D^* is called a *closed topological disk.*

• Let A and B be subsets of the plane, B contained in A, and let ϵ be a positive number. A continuous function f on A into B is an *ϵ-retract of A onto B* if and only if $f(x) = x$ for all $x \in B$ and $|f(x) - x| < \epsilon$ for all $x \in A$.

We now define three properties for an open topological disk G :

(I). There exists a uniformly continuous homeomorphism of D onto G;

(II). For each positive number ϵ, there exists an ϵ - retract of G onto a closed topological disk;

(III). G has the AFPP

THEOREM 1. (I) *implies* (II), *and* (II) *implies* (III).

PROOF. Let us assume that (I) holds for G, and that ϵ is a positive number. Let h be a uniformly continuous homeomorphism of D onto G. There exists a positive number δ which is less than 1 and which has the property that if $|z - w| < \delta$ then $|h(z) - h(w)| < \epsilon$. We define a δ - retract ϕ of D onto a closed disk by letting:

$$\phi(z) = z, \quad \text{if } |z| \leqslant 1 - \delta,$$

$$\phi(z) = (1 - \delta)\, z / |z|, \quad \text{if } 1 - \delta < |z| < 1.$$

It is easy to verify that $h \phi h^{-1}$ is an ϵ - retract of G onto a closed topological disk. Thus (I) implies (II).

Let us assume that (II) holds for G, that f is a continuous function on G into G, and that ϵ is a positive number. We let ψ be an ϵ - retract of G onto a closed topological disk K. If we define $g(x) = \psi(f(x))$ for each $x \in K$, then we obtain a continuous function g on K into K. By the well-known Brouwer Fixed Point Theorem, there exists a point $p \in K$ for which $g(p) = p$. It follows that $|f(p) - p| < \epsilon$, and hence G has the AFPP. Thus (II) implies (III).

COROLLARY. *D has the AFPP.*

PROOF. (I) holds for D since the identity function on D is uniformly continuous. Hence (II) and (III) also hold for D.

THEOREM 2. *If G is a bounded open topological disk whose boundary is locally connected, then G has the AFPP.*

PROOF. By the conformal mapping theorem [1, p. 70] there exists a conformal homeomorphism h of D onto G. E. E. Floyd has shown in [2, p. 658, Corollary 3.2] that h is uniformly continuous. Thus (I) holds for G, and it follows from Theorem 1 that G has the AFPP.

3. Some examples.

EXAMPLE 1. Let S be the set of all points (x, y) in the Cartesian plane for which $0 < x < 1$ and $0 < y < 1$. If $(x, y) \in S$, we define $f(x, y)$ to be the point $(x, xy + \sin(1/x))$. The function f is a homeomorphism which maps S onto an open topological disk S_1. It can be shown that S_1 has property (II) but does not have property (I).

EXAMPLE 2. Let S be as in Example 1. If $(x, y) \in S$, we define $g(x, y)$ to be the point $(x, xy + (1/x) \sin(1/x))$. The function g is a homeomorphism which maps S onto an open topological disk S_2. It can be shown that S_2 has property (III) but does not have property (II).

EXAMPLE 3. Let S_3 be the set of all points with Cartesian coordinates (x, y) for which $x > 1$ and $0 < y < 1$. Let $\rho(x) = x / (1 + x)$ for all $x > 1$. We now define a homeomorphism h on S_3 by letting $h(x, y)$ be the point whose *polar coordinates* are

$$\{(1 - y)\,\rho(x) + y\,\{\rho(x) + \rho(x + 2\pi)\}/2,\ x\}.$$

The function h is a homeomorphism which maps S_3 onto a bounded open topological disk S_4 which does not have property (III).

REMARK. The open topological disk S_2 is not bounded, but by combining the ideas of Examples 2 and 3 it is possible to construct a bounded open topological disk S_5 which has property (III) but does not have property (II). We obtained S_4 by taking an infinite strip in the plane and spiraling it so that it approached the circumference of the unit circle from the inside. To obtain S_5, we start out as in the construction of S_4 and spiral the strip in a circular manner; but after one revolution, we reverse directions and spiral for two revolutions in the opposite direction; we then again reverse directions and spiral for three revolutions, etc.

For (perhaps bounded) open topological disks G, it would be interesting to obtain some conditions which are necessary for the AFPP. For bounded open topological disks whose boundaries are

not locally connected, it would also be of interest to obtain conditions which are sufficient for the AFPP.

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ON A PROBLEM OF HARDY ON THE DISTRIBUTION OF INTEGERS HAVING A GIVEN NUMBER OF PRIME FACTORS III

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CHAPTER 5

Estimation of $\sum_{p \leqslant x^{1-a}} \pi_\nu (x/p)$.

The final result that will be proved in this section is :

If $2 \leqslant \nu \leqslant \nu'$ and (23) holds, then, when $l_2 x \geqslant \nu + 1$,

$$\begin{aligned}\sum_{p \leqslant x^{1-a}} \pi_\nu \left(\frac{x}{p} \right) = {} & \frac{x}{\log x} \sum_{r=0}^{\nu-1} (\nu + 1 - r)\, B_r \frac{(l_2\, x)^{\nu - r}}{(\nu - r)\,!} + \\ & + B \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \frac{(l_2\, x)^{\nu-r-1}}{(\nu - r - 1)\,!} + \\ & + \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^{t-1} \zeta(t) \sum_{r=0}^{\nu-t} B_r \frac{(l_2\, x)^{\nu-r-t}}{(\nu - r - t)\,!} + \\ & + \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \sum_{t=1}^{\nu-r} (-1)^t \frac{(l_2\, x^a)^{\nu-r-t}}{(\nu - r - t)\,!} \sum_{n=1}^{\infty} \frac{a^n}{n^t} - \\ & - \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \frac{(l_2\, x^a)^{\nu-r}}{(\nu - r)\,!} + \\ & + \theta \left\{ c \frac{(l_2\, x)^{\nu+1}}{\nu\,!} \frac{x}{(\log x)^2} h(\nu) \left(1 + \frac{C_{29}}{\nu} \right) \right\}. \qquad (223)\end{aligned}$$

Lemma 41. *If* $1 \leqslant \nu \leqslant l_2\, x$, *then*

$$\sum_{p \leqslant x^{1-a}} \frac{x}{p} \frac{(l_2\, \overline{x/p})^{\nu-1}}{(\nu - 1)\,!} \frac{1}{(\log x - \log p)}$$

$$= (\nu+1)\frac{x}{\log x}\frac{(l_2 x)^\nu}{\nu\,!} + B\frac{x}{\log x}\frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} -$$

$$-\frac{(l_2 x^a)^\nu}{\nu\,!}\frac{x}{\log x} + \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)\,!}\frac{x}{\log x}\log(1-a) +$$

$$+\frac{x}{\log x}\sum_{t=2}^{\nu}(-1)^{t-1}\frac{(l_2 x)^{\nu-t}}{(\nu-t)\,!}\zeta(t) +$$

$$+\frac{x}{\log x}\sum_{t=2}^{\nu}(-1)^{t}\frac{(l_2 x)^{\nu-t}}{(\nu-t)!}\sum_{n=1}^{\infty}\frac{a^n}{n^t} +$$

$$+\theta\left\{C_{30}\frac{(l_2 x)^{\nu+1}}{(\nu+1)\,!}\frac{x}{(\log x)^2}\nu^3\right\}, \tag{224}$$

the empty sums when $\nu = 1$ being regarded as equal to 0.

PROOF. By virtue of (178) and (179), we have

$$\vartheta(x) = x + \epsilon(x), \text{ where } |\epsilon(x)| \leqslant C_{31}\, x/e^{\sqrt{\log x}} \quad \text{for } x \geqslant 2, \tag{225}$$

and

$$\sum_{p\leqslant x}\frac{1}{p} = l_2 x + B + \theta\left\{C_{32}\cdot\frac{1}{e^{\sqrt{\log x}}}\right\}. \tag{226}$$

Let η be chosen such that

$$\log \eta = (l_2 x)^2. \tag{227}$$

We have

$$\sum_{p\leqslant x^{1-a}}\frac{x}{p}\frac{(l_2\, \overline{x/p})^{\nu-1}}{(\nu-1)\,!}\frac{1}{(\log x - \log p)} \leqslant \sum_{2<p\leqslant \eta} + \sum_{\eta\leqslant p\leqslant x^{1-a}}$$

$$= \sum_{2\leqslant p\leqslant \eta} + \sum_{\eta\leqslant p\leqslant x^{1-a}} + \theta\left\{2\frac{x}{(\log x)^2}\frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!}\right\}$$

$$= V_1 + V_2 + \theta\left\{2\frac{x}{(\log x)^2}\frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!}\right\}, \tag{228}$$

say. For $p \leqslant \eta$,

$$(l_2\, \overline{x/p})^{\nu-1} = (l_2 x)^{\nu-1}\left\{1 + \theta\left(C_{33}\frac{\nu}{l_2 x}\frac{(l_2 x)^{\nu-1}}{\log x}\right)\right\};$$

also

$$\sum_{p\leqslant\eta}\frac{1}{p(\log x-\log p)}=\frac{1}{\log x}\sum_{p\leqslant\eta}\frac{1}{p}+\theta\left(\frac{2}{(\log x)^2}\sum_{p\leqslant\eta}\frac{\log p}{p}\right)$$

$$=\frac{1}{\log x}\left\{l_2\,\eta+B+\theta\,\frac{C_{32}}{e^{\sqrt{\log x}}}\right\}+$$

$$+\,\theta\left\{\frac{C_{34}}{(\log x)^2}\log\eta\right\},$$

by virtue of (226) and (113′). Hence by virtue of (227), we get

$$V_1=\frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}\frac{x}{\log x}\left\{1+\theta\left(C_{33}\,\frac{\nu}{l_2\,x}\frac{\log\eta}{\log x}\right)\right\}\times$$

$$\times\left\{l_2\,\eta+B+\theta\left(C_{35}\,\frac{\log\eta}{\log x}\right)\right\}$$

$$=\frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}\frac{x}{\log x}\,(l_2\,\eta+B)+\theta\left\{C_{36}\,\frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}\frac{x}{(\log x)^2}\times\right.$$

$$\left.\times\log\eta\left(1+\nu\frac{l_2\,\eta}{l_2\,x}\right)\right\}$$

$$=\frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}\frac{x}{\log x}\,(l_2\,\eta+B)+\theta\left\{C_{37}\,\frac{x}{(\log x)^2}\frac{(l_2\,x)^{\nu+1}}{(\nu+1)\,!}\,\nu^3\right\}. \quad (229)$$

We now consider V_2. It is easily verified that

$$F(u)=F(u,x)=\frac{(l_2\,\overline{x/u})^{\nu-1}}{(\nu-1)\,!}\,\frac{1}{u\log u\,(\log x-\log u)} \quad (230)$$

is a decreasing function of u for $2\leqslant u\leqslant x^{1-a}$. Also

$$\frac{V_2}{x}=\sum_{\eta\leqslant p\leqslant x^{1-a}}\frac{(l_2\,\overline{x/p})^{\nu-1}}{(\nu-1)\,!}\,\frac{1}{p(\log x-\log p)}$$

$$=\sum_{\eta\leqslant n\leqslant x^{1-a}}\frac{(l_2\,\overline{x/n})^{\nu-1}}{(\nu-1)\,!}\,\frac{1}{n(\log x-\log n)}\cdot\frac{\vartheta(n)-\vartheta(n-1)}{\log n}$$

$$=\sum_{\eta\leqslant n\leqslant x^{1-a}}F(n)+\sum_{\eta\leqslant n\leqslant x^{1-a}}\{\epsilon(n)-\epsilon(n-1)\}\,F(n),$$

$$=W_1+W_2,\ \text{say}. \quad (231)$$

[We use (225), where $F(n) = F(n, x)$ stands for the expression defined in (230)]. Now

$$|W_2| \leqslant \sum_{\eta \leqslant n \leqslant x^{1-a}} \{F(n) - F(n+1)\} \, |\epsilon(n)| +$$

$$+ C_{38} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{\eta \log \eta} \cdot \frac{1}{\log x} \frac{\eta}{e^{\sqrt{\log \eta}}} +$$

$$+ C_{38} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{x^{1-a}} \frac{1}{(\log x)^2} \frac{x^{1-a}}{e^{\sqrt{(1-a)\log x}}}, \qquad \text{by (225),}$$

$$\leqslant C_{39} \sum_{\eta \leqslant n \leqslant x^{1-a}} \frac{n(F(n) - F(n+1))}{e^{\sqrt{\log n}}} +$$

$$+ C_{39} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{(\log x)^2}, \qquad \text{by (225) and (227),}$$

$$\leqslant C_{40} \sum_{\eta \leqslant n \leqslant x^{1-a}} F(n) \left\{ \frac{n}{e^{\sqrt{\log n}}} - \frac{(n-1)}{e^{\sqrt{\log (n-1)}}} \right\} +$$

$$+ C_{40} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{(\log x)^2}$$

$$\leqslant C_{41} \sum_{\eta \leqslant n \leqslant x^{1-a}} \frac{F(n)}{e^{\sqrt{\log n}}} + \frac{C_{41} (l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{(\log x)^2}$$

$$\leqslant C_{42} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{\log x} \sum_{\eta \leqslant n \leqslant x^{1-a}} \frac{1}{n \log n \, e^{\sqrt{\log n}}} +$$

$$+ C_{42} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{(\log x)^2}$$

$$\leqslant C_{43} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{1}{(\log x)^2} \sum_{n=2}^{x} \frac{1}{n \log n}$$

$$\leqslant C_{44} \frac{(l_2 x)^{\nu}}{\nu!} \frac{\nu}{(\log x)^2}. \tag{232}$$

Now

$$W_1 = \sum_{\eta \leqslant n \leqslant x^{1-a}} \frac{(l_2 \overline{x/n})^{\nu-1}}{(\nu-1)!} \frac{1}{n \log n} \frac{1}{(\log x - \log n)}$$

$$= \int_{\eta}^{x^{1-a}} \frac{(l_2\, \overline{x/u})^{\nu-1}}{(\nu-1)!} \frac{du}{u \log u(\log x - \log u)} +$$

$$+ \theta\left(2\, \frac{(l_2\, x)^{\nu-1}}{(\nu-1)!} \frac{1}{(\log x)^2}\right). \tag{233}$$

Also

$$\int_{\eta}^{x^{1-a}} \frac{(l_2\, \overline{x/u})^{\nu-1}}{(\nu-1)!} \frac{du}{u \log u(\log x - \log u)}$$

$$= \int_{x^a}^{x/\eta} \frac{(l_2\, v)^{\nu-1}}{(\nu-1)!} \frac{dv}{v \log v(\log x - \log v)}$$

$$= \frac{1}{\log x} \int_{x^a}^{x/\eta} \frac{(l_2\, v)^{\nu-1}}{(\nu-1)!} \frac{dv}{v \log v} + \frac{1}{\log x} \int_{x^a}^{x/\eta} \frac{(l_2\, v)^{\nu-1}}{(\nu-1)!} \frac{dv}{v(\log x - \log v)}$$

$$= \frac{1}{\log x} \left[\frac{(l_2\, v)^{\nu}}{\nu!}\right]_{x^a}^{x/\eta} + \frac{1}{\log x} \int_{a \log x}^{lx - l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{dw}{\log x - w}$$

$$= \frac{(l_2\, x)^{\nu}}{\nu!} \frac{1}{\log x} - \frac{(l_2\, x^a)^{\nu}}{\nu!} \frac{1}{\log x} + \frac{1}{\log x} \int_{a \log x}^{lx - l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{dw}{\log x - w} +$$

$$+ \theta\left\{C_{45} \frac{(l_2\, x)^{\nu+1}}{(\nu+1)!} \frac{\nu^2}{(\log x)^2}\right\}, \tag{234}$$

since

$$(l_2\, \overline{x/\eta})^{\nu} = (l_2\, x)^{\nu} \left\{1 + \theta\left(C_{46} \frac{\nu}{l_2\, x} \frac{\log \eta}{\log x}\right)\right\}$$

$$= (l_2\, x)^{\nu} + \theta\left\{C_{47} \frac{(l_2\, x)^{\nu+1}}{(\nu+1)!} \frac{\nu^2}{\log x}\right\}.$$

From (233) and (234),

$$W_1 = \frac{(l_2\, x)^{\nu}}{\nu!} \frac{1}{\log x} - \frac{(l_2\, x^a)^{\nu}}{\nu!} \frac{1}{\log x} +$$

$$+ \frac{1}{\log x} \int_{alx}^{lx - l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{dw}{\log x - w} +$$

$$+ \theta\left\{C_{48} \frac{(l_2\, x)^{\nu+1}}{(\nu+1)!} \frac{\nu^2}{(\log x)^2}\right\}. \tag{235}$$

Now

$$\int_{alx}^{lx-l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{dw}{\log x - w}$$

$$= -\frac{1}{(\nu-1)!}\Big[(\log w)^{\nu-1}\log(\log x - w)\Big]_{alx}^{lx-l\eta} +$$

$$+ \int_{alx}^{lx-l\eta} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w}\, dw$$

$$= -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} l_2\eta\left\{1 + \theta\left(C_{49}\frac{\nu}{l_2 x}\frac{l\eta}{lx}\right)\right\} +$$

$$+ \frac{(l_2 x^{\alpha})^{\nu-1}}{(\nu-1)!}\Big\{l_2 x + \log(1-\alpha)\Big\} +$$

$$+ \int_{alx}^{lx-l\eta} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w}\, dw$$

$$= -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} l_2\eta + \frac{(l_2 x^{\alpha})^{\nu-1}}{(\nu-1)!}\Big\{l_2 x + \log(1-\alpha)\Big\} +$$

$$+ \int_{alx}^{lx-l\eta} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w}\, dw +$$

$$+ \theta\left\{C_{50}\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\frac{\nu^3}{\log x}\right\}, \text{ since } l_2\eta\, l\eta \leqslant (l_2 x)^3,$$

$$= -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} l_2\eta + \frac{(l_2 x^{\alpha})^{\nu-1}}{(\nu-1)!}\{l_2 x + \log(1-\alpha)\} +$$

$$+ \int_{alx}^{lx-l\eta} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w}\, dw$$

$$- \int_{lx-l\eta}^{lx} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w}\, dw +$$

$$+ \theta\left\{C_{50}\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\frac{\nu^3}{\log x}\right\}. \qquad (236)$$

Now

$$\left|\int_{lx-l\eta}^{lx} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w}\, dw\right|$$

$$\leqslant \frac{(l_2 x)^{\nu-2}}{(\nu-2)!}\int_{lx-l\eta}^{lx} \frac{|\log(\log x - w)|}{w}\, dw$$

$$\leqslant \frac{(l_2 x)^{\nu-2}}{(\nu-2)!} \frac{2}{\log x} \int_{lx-l\eta}^{lx} \frac{|\log(\log x - w)|}{w} dw$$

$$= \frac{(l_2 x)^{\nu-2}}{(\nu-2)!} \frac{2}{\log x} \log \eta \left\{ l_2 \eta + \int_0^1 |\log z| dz \right\},$$

$$\leqslant C_{52} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{\nu^3}{\log x}, \quad \text{on substituting } \log x - w = z \log \eta.$$

Hence, by (236),

$$\int_{alx}^{lx-l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{dw}{\log x - w}$$

$$= -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} l_2 \eta + \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!} \{ l_2 x + \log(1-a) \} +$$

$$+ \int_{alx}^{lx} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w} dw +$$

$$+ \theta \left\{ C_{53} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{\nu^3}{\log x} \right\}. \qquad (237)$$

Also

$$\int_{alx}^{lx} \frac{(\log w)^{\nu-2}}{(\nu-2)!} \frac{\log(\log x - w)}{w} dw$$

$$= (l_2 x) \int_a^1 \frac{(\log \overline{z \log x})^{\nu-2}}{(\nu-2)!} \frac{dz}{z} + \int_a^1 \frac{(\log \overline{z \log x})^{\nu-2}}{(\nu-2)!} \frac{\log(1-z)}{z} dz$$

$$= \frac{(l_2 x)^{\nu}}{(\nu-1)!} - \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!} l_2 x - \sum_{n=1}^{\infty} \int_a^1 \frac{(\log \overline{z \log x})^{\nu-2}}{(\nu-2)!} \frac{z^{n-1}}{n} dz$$

$$= \frac{(l_2 x)^{\nu}}{(\nu-1)!} - \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!} l_2 x + \sum_{t=2}^{\nu} (-1)^{t-1} \zeta(t) \frac{(l_2 x)^{\nu-t}}{(\nu-t)!} +$$

$$+ \sum_{t=2}^{\nu} (-1)^t \frac{(l_2 x^a)^{\nu-t}}{(\nu-t)!} \sum_{n=1}^{\infty} \frac{a^n}{n^t}, \qquad \text{by } (125'). \qquad (238)$$

Hence from (231), (232), (235) and (237), (238) we get

$$V_2 = (\nu+1) \frac{(l_2 x)^{\nu}}{\nu!} \frac{x}{\log x} + \log(1-a) \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} -$$

$$- \frac{(l_2 x^a)^\nu}{\nu !} \frac{x}{\log x} - \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} l_2 \eta \frac{x}{\log x} +$$

$$+ \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^{t-1} \zeta(t) \frac{(l_2 x)^{\nu-t}}{(\nu-t)!} +$$

$$+ \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^t \frac{(l_2 x^a)^{\nu-t}}{(\nu-t)!} \sum_{n=1}^{\infty} \frac{a_t^n}{n^t} +$$

$$+ \theta \left\{ C_{54} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} \nu^3 \right\}, \tag{239}$$

the empty sums, when $\nu = 1$, being regarded as equal to 0. The lemma follows immediately from (228), (229) and (239) on taking $C_{30} \geqslant 4 + C_{37} + C_{54}$.

LEMMA 42. (223) *holds.*

PROOF.

$$\sum_{p \leqslant x^{1-a}} \pi_\nu(x/p) = \sum_{p \leqslant x^{1-a}} \psi(\nu, x/p) +$$

$$+ \sum_{p \leqslant x^{1-a}} R_\nu(x/p), \quad \text{by virtue of (14).} \tag{240}$$

By virtue of (23) we have, for $p \leqslant x^{1-a}$,

$$|R_\nu(x/p)| \leqslant ch(\nu) \frac{(l_2 \overline{x/p})^\nu}{\nu !} \frac{x}{p(\log x/p)^2}$$

$$\leqslant ch(\nu) \frac{(l_2 x)^\nu}{\nu !} \frac{x}{p(\log \overline{x/p})^2}, \quad \text{if } l_2 x^a \geqslant \nu.$$

Hence, when $l_2 x \geqslant \nu + 1$,

$$\left| \sum_{p \leqslant x^{1-a}} R_\nu(x/p) \right| \leqslant ch(\nu) \frac{(l_2 x)^\nu}{\nu !} \times \sum_{p \leqslant x^{1-a}} \frac{1}{p(\log x - \log p)^2}$$

$$\leqslant ch(\nu) \frac{(l_2 x)^\nu}{\nu !} \frac{x}{(\log x)^2} \times$$

$$\times \left\{ \sum_{p \leqslant x^{1-a}} \frac{1}{p} + \frac{8}{\log x} \sum_{p \leqslant x^{1-a}} \frac{\log p}{p} \right\}, \tag{241}$$

since

$$\frac{1}{(\log x - \log p)^2} = \frac{1}{(\log x)^2} \cdot \frac{1}{(1 - lp/lx)} \cdot \frac{1}{(1 - lp/lx)}$$

$$= \frac{1}{(\log x)^2} \left\{ 1 + \frac{lp}{lx - lp} \right\} \left\{ 1 + \frac{lp}{lx - lp} \right\}$$

$$\leqslant \frac{1}{(\log x)^2} \left\{ 1 + 2 \frac{lp}{lx} \right\}^2$$

$$\leqslant \frac{1}{(\log x)^2} \left\{ 1 + 4 \frac{\log p}{lx} + 4 \frac{(\log p)^2}{(\log x)^2} \right\}$$

$$\leqslant \frac{1}{(\log x)^2} \left\{ 1 + 8 \frac{\log p}{\log x} \right\};$$

hence for (241) we get

$$\left| \sum_{p \leqslant x^{1-\alpha}} R_\nu(x/p) \right| \leqslant c\, h(\nu) \frac{(l_2 x)^\nu}{\nu\,!} \frac{x}{(\log x)^2} \{ l_2 x + C_{55} \}$$

$$\leqslant c\, h(\nu) \frac{(l_2 x)^{\nu+1}}{\nu\,!} \frac{x}{(\log x)^2} \left\{ 1 + \frac{C_{55}}{\nu} \right\}. \qquad (242)$$

Now

$$\sum_{p \leqslant x^{1-\alpha}} \psi(\nu, x/p)$$

$$= \sum_{p \leqslant x^{1-\alpha}} \sum_{r=0}^{\nu-1} B_r \frac{x}{p \log \overline{x/p}} \frac{(l_2 \overline{x/p})^{\nu-r-1}}{(\nu-r-1)\,!}$$

$$= \sum_{r=0}^{\nu-1} B_r \sum_{p \leqslant x^{1-\alpha}} \frac{x}{p\,(\log x - \log p)} \frac{(l_2 \overline{x/p})^{\nu-r-1}}{(\nu - r - 1)\,!}$$

$$= \sum_{r=0}^{\nu-1} B_r\,(\nu + 1 - r) \frac{x}{\log x} \frac{(l_2 x)^{\nu-r}}{(\nu - r)\,!} +$$

$$+ B \frac{x}{\log x} \sum_{r=0}^{\nu-1} \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)\,!} +$$

$$+ \frac{x}{\log x} \sum_{r=0}^{\nu-2} B_r \sum_{t=2}^{\nu-r} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu - r - t)\,!} \zeta(t) +$$

$$+\frac{x}{\log x}\sum_{r=0}^{\nu-2} B_r \sum_{t=2}^{\nu-r}(-1)^t \frac{(l_2 x^a)^{\nu-r-t}}{(\nu-r-t)!}\sum_{n=1}^{\infty}\frac{a^n}{n^t} -$$

$$-\frac{x}{\log x}\sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r}}{(\nu-r)!} +$$

$$+\frac{x}{\log x}\log(1-a)\sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+\theta\left\{C_{30}\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\nu^3 \frac{x}{(\log x)^2}\sum_{r=0}^{\infty}|B_r|\right\}, \qquad (243)$$

by Lemma 41, since $l_2 x \geqslant \nu + 1 \geqslant \nu - r \geqslant 1$ for $0 \leqslant r \geqslant \nu - 1$.

Now

$$\log(1-a)\sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} = -\sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!}\sum_{n=1}^{\infty}\frac{a^n}{n}. \qquad (244)$$

Also

$$C_{30}\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\nu^3 \frac{x}{(\log x)^2}\sum_{r=0}^{\infty}|B_r| \leqslant C.\ C_{56}\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\frac{x}{(\log x)^2}h(\nu). \qquad (245)$$

The lemma follows immediately from (242)-(245) on taking $C_{29} \geqslant C_{55} + C_{56}$, and changing the order of summation in the 3rd and 4th lines of equation (243).

CHAPTER 6

PROOF OF EQUATION 21

Lemma 43.

$$(\nu+1)\,\pi_{\nu+1}(x) = \sum_{r=0}^{\nu}(-1)^r \sum_{p^{r+1}\leqslant x}\pi_{\nu-r}(x/p^{r+1}).$$

PROOF.

$$(\nu+1)\,\pi_{\nu+1}(x) = (\nu+1)\sum_{q_{\nu+1}\leqslant x} 1 = \sum_{p\leqslant x}\sum_{\substack{q_\nu\leqslant x/p\\(p,q_\nu)=1}} 1$$

$$= \sum_{p\leqslant x}\sum_{q_\nu\leqslant x/p} 1 - \sum_{p^2\leqslant x}\sum_{\substack{q_{\nu-1}\leqslant x/p^2\\(q_{\nu-1},p)=1}} 1$$

$$= \sum_{p\leqslant x}\pi_\nu(x/p) - \sum_{p^2\leqslant x}\pi_{\nu-1}(x/p^2) +$$

$$+ \sum_{p^3\leqslant x}\sum_{\substack{q_{\nu-2}\leqslant x/p^3\\(q_{\nu-2},p)=1}} 1.$$

Proceeding thus the result follows.

LEMMA 44. *If* $2\leqslant\nu\leqslant\nu'$ *and* (23) *holds, then when* $l_2 x \geqslant \nu+1$,

$$\sum_{p\leqslant x}\pi_\nu(x/p) = \frac{x}{\log x}\sum_{r=0}^{\nu-1}(\nu+1-r)\,B_r\,\frac{(l_2 x)^{\nu-r}}{(\nu-r)!} +$$

$$+ B_\nu\frac{x}{\log x} + B\frac{x}{\log x}\sum_{r=0}^{\nu-1} B_r\,\frac{(l_2 x)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+ \frac{x}{\log x}\sum_{t=2}^{\nu}(-1)^{t-1}\,\zeta(t)\sum_{r=0}^{\nu-t} B_r\,\frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} +$$

$$+ \theta\left\{C\,\frac{(l_2 x)^{\nu+1}}{\nu!}\,\frac{x}{(\log x)^2}\,h(\nu)\left(1+\frac{C_{57}}{\sqrt{(\nu+1)}}\right)\right\}.$$

PROOF. In Lemma 16 let the P's be the prime numbers p, and Q's be the set of numbers q_ν; $\chi(x) = \psi(x) = 1$ and $\beta = 1-\alpha$. Then we get

$$\sum_{p\leqslant x}\pi_\nu(x/p) = \sum_{p\leqslant x^{1-\alpha}}\pi_\nu(x/p) + \sum_{q_\nu\leqslant x^\alpha}\pi(x/q_\nu) - \pi(x^{1-\alpha})\,\pi_\nu(x^\alpha). \tag{246}$$

Now by virtue of (73) we get

$$\pi(x^{1-\alpha})\,\pi_\nu(x^\alpha) \leqslant C_{58}\,\frac{x^{1-\alpha}}{\log x}\,\frac{x^\alpha}{\log x}\,\frac{(l_2 x + C_{15})^{\nu-1}}{(\nu-1)!}$$

$$\leqslant C_{59}\,\frac{x}{(\log x)^2}\,\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\,h(\nu). \tag{247}$$

From (222), (225), (246) and (247) the result follows immediately, since

$$C_{24}\, g(\nu, x) \leqslant C_{24} \frac{(l_2\, x)^{\nu+1}}{(\nu+1)\,!} \nu^{1/2}\, h(\nu)\,.$$

LEMMA 45. *If* $2 \leqslant \nu \leqslant \nu'$ *and* (23) *holds, then when* $l_2\, x \geqslant \nu + 1$,

$$\sum_{r=1}^{\nu} (-1)^r \sum_{p^r+1 \leqslant x} \pi_{\nu-r}\,(x/p^{r+1})$$

$$= \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^{t-1}\, \lambda(t) \sum_{r=0}^{\nu-t} B_r \frac{(l_2\, x)^{\nu-r-t}}{(\nu-r-t)\,!} +$$

$$+ \theta \left\{ C_{62} \frac{(l_2\, x)^{\nu+1}}{(\nu+1)\,!} h(\nu) \frac{x}{(\log x)^2} \right\}.$$

PROOF.

$$\sum_{r=1}^{\nu} (-1)^r \sum_{p^r+1 \leqslant x} \pi_{\nu-r}\,(x/p^{r+1})$$

$$= \sum_{r=1}^{\nu-1} (-1)^r \sum_{p^r+1 \leqslant x} \pi_{\nu-r}\,(x/p^{r+1}) + (-1)^\nu \sum_{p^\nu+1 \leqslant x} 1$$

$$= \sum_{r=1}^{\nu-1} (-1)^r \sum_{p \leqslant (\log x)^2} \pi_{\nu-r}\,(x/p^{r+1}) +$$

$$+ \sum_{r=1}^{\nu-1} (-1)^r \sum_{(\log x)^{2(r+1)} < p^r+1 \leqslant x/2} \pi_{\nu-r}\,(x/p^{r+1}) +$$

$$+ (-1)^\nu \sum_{p^\nu+1 \leqslant x} 1$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \text{ say.} \tag{248}$$

$$\left|\sum\nolimits_3\right| \leqslant x^{1/\nu+1} \leqslant C_{63} \frac{x}{(\log x)^2} \frac{(l_2\, x)^{\nu+1}}{(\nu+1)\,!} h(\nu). \tag{249}$$

$$\left|\sum\nolimits_2\right| \leqslant \sum_{r=1}^{\nu-1} C_{14} \frac{(l_2\, x + C_{15})^{\nu-r-1}}{(\nu-r-1)\,!} \times$$

$$\times \sum_{(\log x)^{2(r+1)} < p^r+1 \leqslant x/2} (x/p^{r+1})\,(\log \overline{x/p^{r+1}})$$

$$\leqslant C_{64} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \times \sum_{r=1}^{\infty} \sum_{p > (\log x)^2} 1/p^{r+1}$$

$$\leqslant 2C_{64} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} + \sum_{p > (\log x)^2} 1/p^2$$

$$\leqslant C_{65} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2}. \tag{250}$$

From (14) we get, on writing t for r,

$$\sum\nolimits_1 = \sum_{t=1}^{\nu-1} (-1)^t \sum_{p \leqslant (\log x)^2} \psi(\nu - t, x/p^{t+1}) +$$

$$+ \sum_{t=1}^{\nu-1} (-1)^t \sum_{p \leqslant (\log x)^2} R_{\nu-t}(x/p^{t+1}). \tag{251}$$

Now, for $1 \leqslant t \leqslant \nu - 1$, $l_2 x \geqslant \nu + 1 \geqslant 3$, $p \leqslant (\log x)^2$,

$$x/p^{t+1} \geqslant x/(\log x)^{2(l_2 x - 1)} = x/e^{2(l_2 x - 1) l_2 x} \geqslant x^{1/e^2}, \tag{252}$$

since, when $l_2 x \geqslant 3$,

$$2(l_2 x - 1)\, l_2 x < (1 - 1/e^2) \log x. \tag{253}$$

From (252) it follows that for $1 \leqslant t \leqslant \nu - 1$, $p \leqslant (\log x)^2$,

$$l_2 x \geqslant \nu + 1 \geqslant 3,$$

$$l_2 \overline{x/p^{t+1}} \geqslant l_2 x - 2 \geqslant \nu - 1 \geqslant \nu - t \geqslant 1.$$

Hence by (23) we get, if $l_2 x \geqslant \nu + 1 \geqslant 3$, $p \leqslant (\log x)^2$, $\nu \leqslant \nu'$, $1 \leqslant t \leqslant \nu - 1$,

$$| R_{\nu-t}(x/p^{t+1}) | \leqslant C \frac{x}{p^{t+1} (\log \overline{x/p^{t+1}})^2} \frac{(l_2 \overline{x/p^{t+1}})^{\nu-}}{(\nu - t)!} h(\nu - t)$$

$$\leqslant C_{66} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} \frac{h(\nu)}{p^{t+1}}.$$

Hence

$$\left| \sum_{t=1}^{\nu-1} (-1)^t \sum_{p \leqslant (\log x)^2} R_{\nu-t}(x/p^{t+1}) \right|$$

$$\leqslant C_{66} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} h(\nu) \sum_{t=1}^{\infty} \sum_{p} 1/p^{t+1}$$

$$\leqslant C_{67} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} h(\nu). \tag{254}$$

Also

$$\sum_{t=1}^{\nu-1} (-1)^t \sum_{p \leq (\log x)^2} \psi(\nu - t, x/p^{t+1})$$

$$= \sum_{t=1}^{\nu-1} (-1)^t \sum_{r=0}^{\nu-t-1} B_r \sum_{p \leq (\log x)^2} \frac{(l_2\, \overline{x/p^{t+1}})^{\nu-r-t-1}}{(\nu - r - t - 1)!} \times$$

$$\times \frac{x}{p^{t+1}(\log x - \log p^{t+1})}$$

$$= \frac{x}{\log x} \sum_{t=1}^{\nu-1} (-1)^t \sum_{r=0}^{\nu-t-1} B_r \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \sum_{p \leq (l\, x)^2} 1/p^{t+1} +$$

$$+ \theta \left\{ c_{68}\, \nu^2 \frac{l_2 x}{\log x} \quad \frac{x}{\log x} \frac{(l_2 x)^{\nu-2}}{(\nu - 2)!} \left(\sum_{r=0}^{\infty} |B_r| \sum_{t=1}^{\infty} \sum_{p} 1/p^{t+1} \right) \right\}$$

$$= \frac{x}{\log x} \sum_{t=1}^{\nu-1} (-1)^t \sum_{r=0}^{\nu-t-1} B_r \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \lambda(t+1) -$$

$$- \frac{x}{\log x} \sum_{t=1}^{\nu-1} (-1)^t \sum_{r=0}^{\nu-t-1} B_r \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \sum_{p > (\log x)^2} 1/p^{t+1} +$$

$$+ \theta \left\{ C_{68}\, \nu^3 \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} \frac{x}{(\log x)^2} \left(\sum_{r=0}^{\infty} |B_r| \right) \sum_{p} 2/p^2 \right\}$$

$$= \frac{x}{\log x} \sum_{t=1}^{\nu-1} (-1)^t \lambda(t+1) \sum_{r=0}^{\nu-t-1} B_r \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} +$$

$$+ \theta \left\{ \frac{x}{\log x} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} \left(\sum_{r=0}^{\infty} |B_r| \right) \sum_{p > (\log x)^2} \sum_{t=1}^{\infty} 1/p^{t+1} \right\} +$$

$$+ \theta \left\{ C_{69} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} \frac{x}{(\log x)^2} \nu^3 \right\}$$

$$= \frac{x}{\log x} \sum_{t=2} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t} B_r \frac{(l_2 x)^{\nu-r-t}}{(\nu - r - t)!} +$$

$$+ \theta \left\{ \frac{x}{\log x} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} \left(\sum_{r=0}^{\infty} |B_r| \right) . \; 2 \sum_{p > (\log x)^2} 1/p^2 \right\} +$$

$$+ \theta \left\{ C_{69} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} \frac{x}{(\log x)^2} \nu^3 \right\}$$

$$= \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t} B_r \frac{(l_2 x)^{\nu-r-t}}{(\nu - r - t)!} +$$

$$+ \theta \left\{ C_{70} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} h(\nu) \right\}. \tag{255}$$

The lemma follows immediately from (248)–(251), and (254) and (255) on taking

$$C_{62} \geqslant C_{63} + C_{65} + C_{67} + C_{70}.$$

LEMMA 46. *If $2 \leqslant \nu \leqslant \nu'$ and (23) holds, then when $l_2 x \geqslant \nu + 1$,*

$$(\nu+1)\, \pi_{\nu+1}(x) = \frac{x}{\log x} \sum_{r=0}^{\nu} (\nu + 1 - r)\, B_r \frac{(l_2 x)^{\nu-r}}{(\nu - r)!} +$$

$$+ B \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)!} +$$

$$+ \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^{t-1} A(t) \sum_{r=0}^{\nu-t} B_r \frac{(l_2 x)^{\nu-r-t}}{(\nu - r - t)!} +$$

$$+ \theta \left\{ c \frac{(l_2 x)^{\nu+1}}{\nu!} \frac{x}{(\log x)^2} h(\nu + 1) \right\}.$$

PROOF. The dominant term on the right side follows immediately from Lemmas 43-45. Also, by Lemmas 44 and 45 the modulus of the error term does not exceed

$$C \frac{(l_2 x)^{\nu+1}}{\nu!} \frac{x}{(\log x)^2} h(\nu) \left(1 + \frac{C_{57}}{\sqrt{(\nu+1)}} \right) + C_{62} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} h(\nu)$$

$$\leqslant C \frac{(l_2 x)^{\nu+1}}{(\nu)!} \frac{x}{(\log x)^2} h(\nu) \left(1 + \frac{C_{72}}{\sqrt{(\nu+1)}} \right)$$

$$\leqslant C \frac{(l_2 x)^{\nu+1}}{\nu!} \frac{x}{(\log x)^2} h(\nu) \left(1 + \frac{C'}{\sqrt{(\nu+1)}} \right), \text{ since } C' > C_{72} \text{ by (16);}$$

$$= C \frac{(l_2 x)^{\nu+1}}{\nu!} \frac{x}{(\log x)^2} h(\nu + 1).$$

Lemma 47. *If $2 \leqslant \nu \leqslant \nu'$ and (23) holds, then when $l_2 x \geqslant \nu + 1$,*

$$\pi_{\nu+1}(x) = \psi(\nu + 1, x) + \theta \left\{ C \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \frac{x}{(\log x)^2} h(\nu + 1) \right\}.$$

Proof. We have

$$\sum_{r=0}^{\nu} r\, B_r \frac{(l_2 x)^{\nu-r}}{(\nu - r)!} = \sum_{r=1}^{\nu} \frac{(l_2 x)^{\nu-r}}{(\nu-r)!} \left\{ B_{r-1}\, B + \sum_{t=2}^{r} (-1)^{t-1} A(t)\, B_{r-t} \right\}$$

$$= B \sum_{r=0}^{\nu-1} \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)!} +$$

$$+ \sum_{t=2}^{\nu} (-1)^{t-1} A(t) \sum_{r=t}^{\nu} \frac{B_{r-t}(l_2 x)^{\nu-r}}{(\nu - r)!}$$

$$= B \sum_{r=0}^{\nu-1} \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)!} +$$

$$+ \sum_{t=2}^{\nu} (-1)^{t-1} A(t) \sum_{r=0}^{\nu-t} \frac{B_r (l_2 x)^{\nu-r-t}}{(\nu - r - t)!}.$$

From this and from Lemma 46 the result follows immediately.

Theorem. (21) *is true for all ν.*

Proof. Taking $\nu = \nu'$ in Lemma 47 we get, if (23) holds, then when $l_2 x \geqslant \nu + 1$,

$$\pi_{\nu+1}(x) = \psi\{(\nu' + 1)x\} + \theta \left\{ C \frac{(l_2 x)^{\nu'+1}}{(\nu'+1)!} \frac{x}{(\log x)^2} h(\nu' + 1) \right\},$$

or

$$| R_{\nu'+1}(x) | \leqslant C \frac{(l_2 x)^{\nu'+1}}{(\nu'+1)!} \frac{x}{(\log x)^2} h(\nu' + 1).$$

Hence if (21) is true for $1 \leqslant \nu \leqslant \nu'$, then it is also true for the index $\nu' + 1$, provided $\nu' \geqslant 2$. But by virtue of (18) and (19), (21) is true when $\nu = 1, 2$. Hence by induction (21) is true for *all* ν.

ON A PROBLEM OF HARDY ON THE DISTRIBUTION OF INTEGERS HAVING A GIVEN NUMBER OF PRIME FACTORS IV

By L. G. SATHE

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CHAPTER 7

Proof of the main results (5)-(7)

Introduction. From equation (20) which has been proved in the preceeding section we shall first deduce that

If

$$\nu_0 = [\log \log x], \tag{256}$$

then as $x \rightarrow \infty$

$$\pi_{\nu_0}(x) \sim \frac{6}{\pi^2} \frac{1}{\sqrt{(2\pi)}} \frac{x}{\sqrt{(\log \log x)}}, \tag{257}$$

$$\sigma_{\nu_0}(x) \sim \rho_{\nu_0}(x) \sim \frac{1}{\sqrt{(2\pi)}} \frac{x}{\sqrt{(\log \log x)}}. \tag{258}$$

The more general results are proved afterwards in Chapter 8.

From the equation (20) we see that it is necessary, in the first instance, to find an asymptotic formula for $\psi(\nu, x)$ when ν is a function of x. (257) then follows immediately. (258) is deduced by connecting $\sigma_\nu(x)$ and $\rho_\nu(x)$ with $\pi_\nu(x)$ by algebraic identities.

Throughout this section the following notations will be used :

$a_1, a_2, \ldots$ *denote absolute positive constants,*

$\omega = \nu / l_2 x$; $f(z)$ *denotes the integral function defined by equations* (28) *and* (29), *viz:*

$$f(z) = \sum_{r=0}^{\infty} B_r z^r = \frac{1}{\Gamma(z+1)} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{e^{1/p}} \right\}^z \prod_p \left\{ \left(1 + \frac{z}{p}\right) e^{-z/p} \right\},$$

$F(n)$ = total number of prime factors of n,

$t(n)$ = number of different prime factors of n,

$F(1) = t(1) = 0.$

It is easily verified that

$$\frac{1}{10} \leqslant f(z) \leqslant 10, \quad \text{for } 0 \leqslant z \leqslant 1. \tag{259}$$

LEMMA 48. *If $|z| \leqslant 10$, $|\eta| \leqslant 1000$, then*

$$f(z+\eta) = f(z) + \theta\{a_1 |\eta|\}. \tag{260}$$

PROOF.

$$|f(z+\eta) - f(z)| = \left| \sum_{r=1}^{\infty} B_r \{(z+\eta)^r - z^r\} \right|$$

$$\leqslant |\eta| \sum_{r=1}^{\infty} |B_r| r (|z| + |\eta|)^{r-1}, \text{ as in } (100')$$

$$\leqslant |\eta| \sum_{r=1}^{\infty} |B_r| r (1010)^{r-1+1}$$

$$\leqslant a_1 |\eta|, \text{ by virtue of Lemma 2.}$$

LEMMA 49. *If $1 \leqslant \nu \leqslant l_2 x$ then*

$$\psi(\nu, x) = \left\{ 1 + \theta\left(\frac{a_2}{l_2 x}\right) \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}. \tag{261}$$

PROOF. If $r \leqslant \nu^{1/3} + 1$, $\nu \geqslant 3$ then by Stirling's theorem :

$$\frac{(\nu-1)!}{(\nu-r-1)!} = \nu^r \left\{ 1 + \theta\left(a_3 \frac{r^2}{\nu}\right) \right\}, \tag{262}$$

and hence,

$$\frac{(l_2 x)^{\nu-r-1}}{(\nu-r-1)!} = \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \left(\frac{\nu}{l_2 x}\right)^r \left\{ 1 + \theta\left(a_3 \frac{r^2}{\nu}\right) \right\}. \tag{263}$$

It is evident that for $\nu = 1, 2$

$$\psi(\nu, x) = \left\{ 1 + \theta\left(\frac{a_4}{l_2 x}\right) \right\} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \frac{x}{\log x}$$

$$= \left\{ 1 + \theta\left(\frac{a_5}{l_2 x}\right) \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \frac{x}{\log x}, \qquad (264)$$

since, $1 = f(0) = \left\{ 1 + \theta\left(\frac{a_6}{l_2 x}\right) \right\} f(\omega)$, for $\nu = 1, 2,$ by virtue of Lemma 48 and equation (259).

Now, for $\nu \geqslant 3$, $1 \leqslant r' = [\nu^{1/3}] \leqslant \nu - 2$. Hence,

$$\frac{\psi(\nu, x)}{x/\log x} = \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)!}$$

$$= \sum_{r=0}^{r'} + \sum_{r=r'+1}^{\nu-1} = \sum\nolimits_1 + \sum\nolimits_2, \text{ say.} \qquad (265)$$

$$\left|\sum\nolimits_2\right| \leqslant \sum_{r=r'+1}^{\nu-1} |B_r| \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)!}$$

$$\leqslant \frac{(l_2 x)^{\nu-2}}{(\nu - 2)!} \sum_{r=r'+1}^{\infty} |B_r|, \qquad \text{since } \nu \leqslant l_2 x,$$

$$\leqslant a_7 \frac{(l_2 x)^{\nu-2}}{(\nu - 2)!} \sum_{r=r'+1}^{\infty} \frac{1}{2^r}, \qquad \text{by Lemma 2,}$$

$$\leqslant a_8 \frac{(l_2 x)^{\nu-2}}{(\nu - 1)!} f(\omega), \text{ since } r' \geqslant \nu^{1/3} - 1 \text{ and } f(\omega) \geqslant \frac{1}{10}. \qquad (266)$$

Also by virtue of (263)

$$\sum\nolimits_1 = \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \sum_{r=0}^{r'} B_r \left(\frac{\nu}{l_2 x}\right)^r + \theta \left\{ a_3 \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \frac{\nu}{l_2 x} \sum_{r=1}^{\infty} |B_r| \frac{r^2}{\nu} \right\},$$

$$\text{since } \left(\frac{\nu}{l_2 x}\right)^{r-1} \leqslant 1 \text{ for } r \geqslant 1,$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \sum_{r=0}^{\infty} B_r \left(\frac{\nu}{l_2 x}\right)^r - \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \sum_{r=r'+1}^{\infty} B_r \left(\frac{\nu}{l_2 x}\right)^r +$$

$$+ \theta \left\{ a_9 \frac{(l_2 x)^{\nu-2}}{(\nu - 1)!} \right\}$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} f(\omega) + \theta \left\{ \frac{(l_2 x)^{\nu-2}}{(\nu-1)!} \nu \sum_{r=r'+1}^{\infty} |B_r| \right\} + \theta \left\{ a_9 \frac{(l_2 x)^{\nu-2}}{(\nu-1)!} \right\}$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} f(\omega) \left\{ 1 + \theta \left(\frac{a_{10}}{l_2 x} \right) \right\}, \text{ as in (266).} \tag{267}$$

The lemma follows from (264)-(267), taking $a_2 \geqslant a_5 + a_8 + a_{10}$.

LEMMA 50. *If* $1 \leqslant \nu \leqslant l_2 x$, *then*

$$\pi_\nu(x) = \left\{ 1 + \theta \left(\frac{a_{11}}{l_2 x} \right) \right\} f(\omega) \frac{(l_2 x)^{\nu-1} x}{(\nu-1)! \log x}. \tag{268}$$

Hence by Lemma 48 *and Stirling's theorem, if* $\nu_0 = [\log \log x]$

$$\pi_{\nu_0}(x) \sim \frac{6}{\pi^2} \cdot \frac{1}{(2\pi)^{1/2}} \cdot \frac{1}{(\log \log x)^{1/2}}$$

since $f(\omega) \rightarrow f(1) = 6/\pi^2$.

PROOF. If $\nu \leqslant l_2 x$ we have

$$h(\nu) = \prod_{m=1}^{\nu} \left(1 + \frac{c'}{\sqrt{m}}\right) \leqslant a_{12} \frac{\log x}{(l_2 x)^2}. \tag{269}$$

Hence from (20), (261) and (269) we get

$$\pi_\nu(x) = \left\{ 1 + \theta \left(\frac{a_2}{l_2 x} \right) \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} +$$

$$+ \theta \left\{ c\, a_{12} \frac{x}{\log x} \frac{(l_2 x)^{\nu-2}}{(\nu-1)!} \right\}.$$

From this the result follows immediately by taking $a_{11} \geqslant a_2 + 10 c a_{12}$ and using equation (259).

WE NOW CONSIDER $\sigma_\nu(x)$.

LEMMA 51.

$$\sigma_\nu(x) = \sum_{\substack{m \\ 0 \leqslant F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}(x/m^2). \tag{270}$$

PROOF. Every $n \leqslant x$ for which $F(n) = \nu$ is *uniquely* representable in the form

$$n = m^2 q_{\nu - F(m^2)} \leqslant x; \; F(m^2) \leqslant \nu, \tag{271}$$

with the convention $q_0 = 1$.

From (271) the lemma follows on summing with respect to m.

LEMMA 52. *If* $2 \leqslant \nu \leqslant l_2 x$, *then*

$$\sum_{\substack{m^2 > 2(\nu^{\frac{1}{3}}+1) \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}(x/m^2) \leqslant a_{13} \frac{(l_2 x)^{\nu-2} x}{(\nu-1)! \log x}. \tag{272}$$

PROOF. For $m^2 > 4$ we have $F(m^2) > 2$. Hence, if $\nu \geqslant 3$

$$\sum_{\substack{m^2 > 2(\nu^{\frac{1}{3}}+1) \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}(x/m^2) \leqslant \sum_{\substack{x/2 \geqslant m^2 \geqslant 2^{\nu^{\frac{1}{3}}} \\ 2 \leqslant F(m^2) \leqslant \nu - 1}} \pi_{\nu - F(m^2)}(x/m^2) + \sum_{m^2 \leqslant x} 1$$

$$\leqslant a_{14} \times \sum_{\substack{x/2 \geqslant m^2 \geqslant 2^{\nu^{\frac{1}{3}}} \\ 2 \leqslant F(m^2) \leqslant \nu - 1}} \frac{(l_2\, \overline{x/m^2} + a_{15})^{\nu - F(m^2) - 1}}{(\nu - F(m^2) - 1)!} \frac{1}{m^2 \log \overline{x/m^2}} + \sqrt{x},$$

by equation (73),

$$\leqslant a_{14} \frac{(l_2 x + a_{15})^{\nu-2}}{(\nu-2)!} \times \sum_{x/2 \geqslant m^2 \geqslant 2^{\nu^{\frac{1}{3}}}} \frac{1}{m^2 \log \overline{x/m^2}} + \sqrt{x}, \tag{273}$$

since $3 \leqslant \nu \leqslant l_2 x$ and $\log\log 2 + a_{15} > 0$.

Also

$$\sum_{x/2 \geqslant m^2 \geqslant 2^{\nu^{\frac{1}{3}}}} \frac{1}{m^2 \log \overline{x/m^2}} \leqslant \frac{2}{\log x} \sum_{m^2 \geqslant 2^{\nu^{\frac{1}{3}}}} \frac{1}{m^2} + \frac{1}{\log 2} \sum_{m^2 \geqslant \sqrt{x}}^{\infty} \frac{1}{m^2}$$

$$\leqslant \frac{a_{16}}{\log x} \frac{1}{\nu}. \tag{274}$$

From (273)-(274) it follows that for $3 \leqslant \nu \leqslant l_2 x$ the sum on the left of (272) is not greater than $a_{17} \dfrac{(l_2 x)^{\nu-2}}{(\nu-1)!} \dfrac{x}{\log x}$. When $\nu = 2$ the sum does not exceed $\sqrt{x} \leqslant \dfrac{x}{\log x}$ for $x \geqslant e^{e^2}$.

Hence the lemma.

LEMMA 53. *If* $2 \leqslant \nu \leqslant l_2 x$, *then*

$$\pi_\nu(x) + \sum_{\substack{4 \leqslant m^2 \leqslant 2(\nu^{\frac{1}{3}}+1) \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}(x/m^2)$$

$$= \left\{ \prod_p \left(1 - \frac{\omega^2}{p^2} \right)^{-1} \right\} f(\omega) \frac{(l_2\, x)^{\nu-1}}{(\nu-1)\,!} \frac{x}{\log x} \left\{ 1 + \theta \left(a_{18} \frac{\nu^{2/3}}{l_2\, x} \right) \right\}. \tag{275}$$

PROOF. We have $2^{F(n)} \leqslant n$. Hence, for

$$4 \leqslant m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)}, \tag{276}$$

$$2 \leqslant F(m^2) \leqslant [\nu^{1/3}] + 1. \tag{277}$$

Also for $m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)}$, $l_2\, x \geqslant \nu \geqslant 3$,

$$\left.\begin{aligned} & x/m^2 \geqslant \sqrt{x}, \\ & l_2\, \overline{x/m^2} \geqslant \nu - F(m^2) \geqslant 1, \text{ by } (277) \\ & \frac{(l_2\, \overline{x/m^2})^{\nu - F(m^2)-1}}{\log \overline{x/m^2}} = \frac{(l_2\, x)^{\nu - F(m^2)-1}}{\log x} \left\{ 1 + \theta \left(a_{19} \frac{l_2\, x}{\log x} \right) \right\} \end{aligned}\right\}, \tag{278}$$

$$l_2\, \overline{x/m^2} = l_2\, x \left\{ 1 + \theta \left(\frac{4}{\log x} \right) \right\}. \tag{279}$$

Hence from (268), (277) and (278) we get

$$\sum_{\substack{4 \leqslant m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)} \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}\,(x/m^2) = \sum_{4 \leqslant m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)}} \pi_{\nu - F(m^2)}\,(x/m^2)$$

$$= \sum_{4 \leqslant m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)}} \left\{ 1 + \theta \left(\frac{a_{11}}{l_2\, \overline{x/m^2}} \right) \right\} f \left(\frac{\nu - F(m^2)}{l_2\, \overline{x/m^2}} \right) \times$$

$$\times \frac{(l_2\, \overline{x/m^2})^{\nu - F(m^2)-1}}{(\nu - F(m^2) - 1)\,!} \frac{x}{m^2 \log \overline{x/m^2}}$$

$$= \left\{ 1 + \theta \left(\frac{a_{20}}{l_2 x} \right) \right\} \frac{x}{\log x} \sum_{4 \leqslant m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)}} f \left(\frac{\nu - F(m^2)}{l_2\, \overline{x/m^2}} \right) \times$$

$$\times \frac{(l_2\, x)^{\nu - F(m^2)-1}}{(\nu - F(m^2) - 1)\,!} \frac{1}{m^2}. \tag{280}$$

Now

$$\frac{\nu - F(m^2)}{l_2\, \overline{x/m^2}} - \frac{\nu}{l_2\, x} = \frac{\nu}{l_2\, \overline{x/m^2}} - \frac{\nu}{l_2\, x} + \theta \left(4 \frac{\nu^{1/3}}{l_2\, x} \right)$$

$$= \theta \left(12 \frac{\nu^{1/3}}{l_2\, x} \right), \qquad \text{by (279).}$$

Hence by (260)

$$f\left(\frac{\nu - F(m^2)}{l_2\, x/m^2}\right) = f\left(\frac{\nu}{l_2\, x}\right) + \theta\left(12\, a_1 \frac{\nu^{1/3}}{l_2\, x}\right) = f(\omega) + \theta\left(a_{21} \frac{\nu^{1/3}}{l_2\, x}\right). \tag{281}$$

From (280) and (281) we get, since $f(\omega) \geqslant 1/10$,

$$\sum_{4 \leqslant m^2 \leqslant 2^{\nu^{\frac{1}{3}}}+1} \pi_{\nu - F(m^2)}(x/m^2)$$

$$= \left\{1 + \theta\left(a_{22} \frac{\nu^{1/3}}{l_2 x}\right)\right\} f(\omega) \frac{x}{\log x} \sum_{4 \leqslant m^2 \leqslant 2^{\nu^{\frac{1}{3}}}+1} \frac{(l_2\, x)^{\nu - F(m^2) - 1}}{(\nu - F(m^2) - 1)!} \frac{1}{m^2}. \tag{282}$$

Also by (263) and (277) if $\nu \geqslant 3$

$$\sum_{4 \leqslant m^2 \leqslant 2^{\nu^{\frac{1}{3}}}+1} \frac{(l_2\, x)^{\nu - F(m^2) - 1}}{(\nu - F(m^2) - 1)!} \frac{1}{m^2}$$

$$= \frac{(l_2\, x)^{\nu - 1}}{(\nu - 1)!} \sum_{4 \leqslant m^2 \leqslant 2^{\nu^{\frac{1}{3}}}+1} \frac{\omega\, F(m^2)}{m^2} +$$

$$+ \theta\left\{a_3 \frac{(l_2\, x)^{\nu - 1}}{(\nu - 1)!} \frac{4\, \nu^{2/3}}{\nu}\, \omega . \sum_{m=2}^{\infty} \frac{1}{m^2}\right\}, \text{ since } |\omega| \leqslant 1,$$

$$= \frac{(l_2\, x)^{\nu - 1}}{(\nu - 1)!} \sum_{m=2}^{\infty} \frac{\omega^{F(m^2)}}{m^2} + \theta\left\{a_{24} \frac{\nu^{2/3}}{l_2\, x} \frac{(l_2\, x)^{\nu - 1}}{(\nu - 1)!}\right\}, \tag{283}$$

since

$$\sum_{m^2 > 2^{\nu^{\frac{1}{3}}}+1} \frac{\omega^{F(m^2)}}{m^2} \leqslant \omega \sum_{m^2 > 2^{\nu^{\frac{1}{3}}}} \frac{1}{m^2} < \frac{100}{l_2\, x}.$$

From (268), (282) and (283) we get, if $l_2\, x \geqslant \nu \geqslant 3$,

$$\pi_\nu(x) + \sum_{\substack{4 \leqslant m^2 \leqslant 2^{\nu^{\frac{1}{3}}}+1 \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m)^2}(x/m^2)$$

$$= \left\{1 + \theta\left(a_{23} \frac{\nu^{2/3}}{l_2\, x}\right)\right\} f(\omega) \frac{(l_2\, x)^{\nu - 1}}{(\nu - 1)!} \frac{x}{\log x} \times$$

$$\times \left\{1 + \sum_{m=2}^{\infty} \frac{\omega^{F(m^2)}}{m^2} + \theta\left(a_{24} \frac{\nu^{2/3}}{l_2\, x}\right)\right\},$$

taking $a_{23} \geqslant a_{22} + a_{11}$,

$$= 1 + \theta\left(a_{23}\frac{\nu^{2/3}}{l_2 x}\right)\Big\} f(\omega)\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}\frac{x}{\log x} \times$$

$$\times \left\{\sum_{m=1}^{\infty}\frac{\omega^{F(m^2)}}{m^2}\right\}\left\{1 + \theta\left(a_{24}\frac{\nu^{2/3}}{l_2 x}\right)\right\}$$

$$= \left\{1 + \theta\left(a_{25}\frac{\nu^{2/3}}{l_2 x}\right)\right\} f(\omega)\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}\frac{x}{\log x}\sum_{m=1}^{\infty}\frac{\omega^{F(m^2)}}{m^2}. \tag{284}$$

From (284) the lemma follows, since by equation (10), page 16 of Ingham's tract

$$\sum_{m=1}^{\infty}\frac{\omega^{F(m^2)}}{m^2} = \prod_p\left(1 - \frac{\omega^2}{p^2}\right)^{-1}$$

and since (275) is evident when $\nu = 2$ with a suitable value of a_{18}.

Lemma 54. *If* $1 \leqslant \nu \leqslant l_2 x$, *then*

$$\sigma_\nu(x) = \left\{1 + \theta\left(a_{26}\frac{\nu^{2/3}}{l_2 x}\right)\right\} f(\omega)\left\{\prod_p\left(1 - \frac{\omega^2}{p^2}\right)^{-1}\right\}\frac{x}{\log x}\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}. \tag{285}$$

In particular,

$$\sigma_{\nu_0}(x) \sim \frac{1}{\sqrt{(2\pi)}}\frac{x}{\sqrt{(\log\log x)}} \sim \frac{\pi^2}{6}\pi_{\nu_0}(x). \tag{286}$$

Proof. By Lemma 51

$$\sigma_\nu(x) = \pi_\nu(x) + \sum_{\substack{4 \leqslant m^2 \leqslant 2^{(\nu^{\frac{1}{3}}+1)} \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}(x/m^2) + \sum_{\substack{m^2 \geqslant 2^{(\nu^{\frac{1}{3}})} \\ F(m^2) \leqslant \nu}} \pi_{\nu - F(m^2)}(x/m^2). \tag{287}$$

From Lemmas 52 and 53 and equation (259), (285) follows for $2 \leqslant \nu \leqslant l_2 x$ if $a_{26} \geqslant a_{18} + 10a_{13}$. Also, (285) is true when $\nu = 1$ with the proper value of a_{26}.

(286) follows from (285) by using (260) and applying Stirling's theorem

This completes the investigation of $\sigma_\nu(x)$ for $1 \leqslant \nu \leqslant l_2 x$.

WE CONSIDER $\rho_\nu(x)$

LEMMA 55. *Let $\pi'_\nu(x, m)$ denote the number of*

$$q_\nu \leqslant x \text{ with } (q_\nu, m) = 1. \tag{288}$$

Then

$$\rho_\nu(x) = \sum_{\substack{m \\ 0 \leqslant t(m) \leqslant \nu}} \sum_{d|m} |\mu(d)| \, \pi'_{\nu-t(m)}(x/dm^2, m), \tag{289}$$

where $\mu(n)$ is the Möbius function and $\pi'_0(x, m) = \pi_0(x)$.

PROOF. If $n \leqslant x$ and $t(n) = \nu$ then n can be represented in the form

$$n = m^2 d \, q_{\nu - t(m)} \leqslant x, \ (q_{\nu-t(m)}, m) = 1, \tag{290}$$

where d is a square free divisor of m. It may be noted that $t(n) = t(n^2)$. Further the representation in the form (290) is unique for a given n — the numbers m, d and $q_{\nu-t(m)}$ being completely determined by it. The contribution to $\rho_\nu(x)$ by a fixed m for which $t(m) \leqslant \nu$ is, therefore, equal to

$$\sum_{d|m} |\mu(d)| \sum_{\substack{q_{\nu-t(m)} \leqslant x/m^2 d \\ (q_{\nu-t(m)}, m) = 1}} 1 = \sum_{d|m} |\mu(d)| \, \pi'_{\nu-t(m)}(x/m^2 d, m).$$

From this the lemma follows immediately on summing with respect to those m for which $t(m) \leqslant \nu$.

LEMMA 56. *If $p_1, \ldots, p_j$ are different primes, then*

$$\pi'_\nu(x, p_1, p_2, \ldots, p_j) = \sum_{\substack{0 \leqslant \Sigma a_i \leqslant \nu \\ a_j \geqslant 0}} (-1)^{\Sigma a_i} \, \pi_{\nu - \Sigma a_i}(x/p_1^{a_1} \cdots p_j^{a_j}) \tag{291}$$

for $j \geqslant 1$.

PROOF. We have obviously for *any* prime p

$$\pi'_\nu(x, p) = \sum_{0 \leqslant a \leqslant \nu} (-1)^a \, \pi_{\nu-a}(x/p^a). \tag{292}$$

Also

$$\pi'_\nu(x, p_1 \cdots p_j) = \sum_{0 \leqslant a_j \leqslant \nu} (-1)^{a_j} \, \pi'_{\nu-a_j}(x/p_j^{a_j}, p_1 \cdots p_{j-1}). \tag{293}$$

By (223) it follows that (291) is true for $j = 1$. Assuming that (291) is true for the index $j - 1 \geqslant 1$ we show that it is true for j with the help of (293). Hence the lemma follows by induction through j.

Lemma 57. *If $p_1, \ldots, p_j$ are any different primes such that*

$$p_1 p_2 \cdots p_j \leqslant 2^{\nu^{1/12}+1} \tag{294}$$

then if

$$l_2 x \geqslant \nu \geqslant 2$$

$$\pi'_\nu(x, p_1 \ldots p_j) = \left\{ 1 + \theta\left(a_{27} \frac{\nu^{5/6}}{l_2 x}\right) \right\} \times$$

$$\times f(\omega) \left\{ \prod_{p | p_1 p_2 \cdots p_j} \left(1 + \frac{\omega}{p}\right)^{-1} \right\} \frac{x}{\log x} \frac{(l_2 x)^{\nu - 1}}{(\nu - 1)!}.$$

It may be remarked explicitly that a_{27} is an absolute constant and, as such, is independent of $p_1, \ldots, p_j$; ν or x.

Proof. By Lemma 56 we have,

$$\pi'_\nu(x, p_1 \ldots p_j) = \pi_\nu(x) + \sum_{1 \leqslant \Sigma a_i \leqslant \nu^{\frac{1}{3}}} (-1)^{\Sigma a_i} \pi_{\nu - \Sigma a_i}(x / p_1^{a_1} \ldots p_j^{a_j}) +$$

$$+ \sum_{\nu^{\frac{1}{3}} < \Sigma a_i \leqslant \nu} (-1)^{\Sigma a_i} \pi_{\nu - \Sigma a_i}(x / p^{a_1} \ldots p_j^{a_j})$$

$$= \pi_\nu(x) + S_1 + S_2, \text{ say.} \tag{295}$$

$$|S_2| \leqslant \sum_{\nu^{1/3} \leqslant \Sigma a_i \leqslant \nu} \pi_{\nu - \Sigma a_i}(x / p_1^{a_1} \ldots p_j^{a_j})$$

$$\leqslant \sum_{\nu^{1/3} \leqslant \Sigma a_i \leqslant \nu - 1} \pi_{\nu - \Sigma a_i}(x / p_1^{a_1} \ldots p_j^{a_j}) + \sum_{\Sigma a_i = \nu} 1.$$

Now by (294), $p_1^{a_1} \ldots p_j^{a_j} \leqslant (p_1 \cdots p_j)^{\nu - 1} \leqslant \sqrt{x}$ when $l_2 x \geqslant \nu \geqslant 2$.

Hence by (73), noting that $\nu \leqslant l_2 x$, we get

$$|S_2| \leqslant a_{28} \frac{x}{\log x} \frac{(l_2 x)^{\nu - 2}}{(\nu - 2)!} \sum_{\Sigma a_i \geqslant \nu^{1/3}} \frac{1}{p_1^{a_1} \ldots p_j^{a_j}} + (\nu + 1)^j. \tag{296}$$

Now in every term of $\sum_2$ there is an $a_i \geqslant \frac{\nu^{1/3}}{j} \geqslant \frac{\nu^{1/6}}{2}$, since by (294),

$$j \leqslant \nu^{1/12} + 1 \leqslant 2\nu^{1/12}. \qquad (296')$$

Hence

$$\sum_{\Sigma a_j \geqslant \nu^{1/3}} \frac{1}{p_1^{a_1} \dots p_j^{a_j}} \leqslant \left\{ \sum_{a_i \geqslant 0}^{\infty} \frac{1}{\Sigma 2^{a_2} 3^{a_3} \dots j^{a_j}} \right\} \times$$

$$\times \left\{ \sum_{a_1 \geqslant (\nu^{1/6})/2 \,..}^{\infty} \frac{1}{2^a} + \dots + \sum_{a_j \geqslant (\nu^{1/6})/2}^{\infty} 1/2^{a_j} \right\}$$

$$\leqslant \left(\sum_{a=0}^{\infty} \frac{1}{2^a} \right) \cdots \left(\sum_{a=0}^{\infty} \frac{1}{j^a} \right) j \, \frac{2}{2^{(\nu^{1/6})/2}}$$

$$\leqslant \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{j}{j-1} \, j \, \frac{2}{2^{(\nu^{1/6})/2}}$$

$$\leqslant \frac{2j^2}{2^{(\nu^{1/6})/2}} \leqslant \frac{2^{40}}{\nu}, \qquad (297)$$

by virtue of (296′). From (296) and (297) we get

$$|S_2| \leqslant a_{29} \frac{x}{\log x} \frac{(l_2 x)^{\nu-2}}{(\nu-1)!}. \qquad (298)$$

Now for $1 \leqslant a_1 + a_2 + \dots + a_j \leqslant \nu^{1/3}$, $p_1 p_2 \dots p_j \leqslant 2^{\nu^{1/12}+1}$, $l_2 x \geqslant \nu \geqslant 2$ we have :

$$l_2 (x/p_1^{a_1} \dots p_j^{a_j}) \geqslant l_2 x - \log 2 \geqslant \nu - a_1 - \dots - a_j \geqslant 1,$$

$$l_2 (x/p_1^{a_1} \dots p_j^{a_j}) = l_2 x \left\{ 1 + \theta\left(4 \frac{\nu^{1/3}}{\log x} \right) \right\}.$$

Hence, as in (281)

$$f\left(\frac{\nu - a_1 - \dots - a_j}{l_2 (x/p_1^{a_1} \dots p_j^{a_j})} \right) = f(\omega) \left\{ 1 + \theta \left(a_{30} \frac{\nu^{1/3}}{l_2 x} \right) \right\}$$

$$\frac{(l_2 x/p_1^{a_1} \,.\, p_j^{a_j})^{\nu - \Sigma a_i - 1}}{\log (x/p_1^{a_1} \dots p_j^{a_j})} = \frac{(l_2 x)^{\nu - \Sigma a_i - 1}}{\log x} \times$$

$$\times \left\{ 1 + \theta \left(a_{31} \frac{(l_2 x)^2}{\log x} \right) \right\}$$

From this and Lemma 50 we get

$$S_1 = \sum_{1 \leq \Sigma a_i \leq \nu^{\frac{1}{3}}} (-1)^{\Sigma a_i} f\left(\frac{\nu - a_1 \dots - a_j}{l_2 x / p_1^{a_1} \dots p_j^{a_j}}\right) \times$$

$$\times \frac{(l_2 x / p_1^{a_1} \dots p_j^{a_j})^{\nu - \Sigma a_i - 1}}{(\nu - a_1 - \dots - a_j - 1)!} \frac{x}{p_1^{a_1} \dots p_j^{a_j} \log(x / p_1^{a_1} \dots p_j^{a_j})} +$$

$$+ \theta \left\{ \frac{a_{32}}{l_2 x} \frac{(l_2 x)^{\nu - 2}}{(\nu - 2)!} \frac{x}{\log x} \sum_{1 \leq \Sigma a_i} 1/p_1^{a_1} \dots p_j^{a_j} \right\},$$

since $1/10 \leq f(z) \leq 10$ for $0 \leq z \leq 1$,

$$= f(\omega) \frac{x}{\log x} \left\{ \sum_{1 \leq \Sigma a_i \leq \nu^{\frac{1}{3}}} (-1)^{\Sigma a_i} \frac{(l_2 x)^{\nu - \Sigma a_i - 1}}{(\nu - a_1 - \dots - a_j - 1)!} \times \right.$$

$$\left. \times \frac{1}{p_1^{a_1} \dots p_j^{a_j}} \right\} +$$

$$+ \theta \left\{ a_{33} \frac{\nu^{1/3}}{l_2 x} f(\omega) \frac{(l_2 x)^{\nu - 2}}{(\nu - 2)!} \frac{x}{\log x} \sum_{a_i \geq 0}^{\infty} \frac{1}{2^{a_1} \dots (j+1)^{a_j}} \right\}$$

$$= f(\omega) \frac{x}{\log x} \left\{ \sum_{1 \leq \Sigma a_i \leq \nu^{\frac{1}{3}}} (-1)^{\Sigma a_i} \frac{(l_2 x)^{\nu - \Sigma a_i - 1}}{(\nu - a_1 - \dots - a_j - 1)!} \times \right.$$

$$\left. \times \frac{1}{p_1^{a_1} \dots p_j^{a_j}} \right\} + \theta \left\{ a_{34} \frac{\nu^{1/2}}{l_2 x} f(\omega) \frac{(l_2 x)^{\nu - 2}}{(\nu - 2)!} \frac{x}{\log x} \right\}, \qquad (299)$$

since as in (297),

$$\sum_{a_i \geq 0}^{\infty} \frac{1}{2^{a_1} \dots (j+1)^{a_j}} \leq j + 1 \leq 4 \nu^{1/12} \text{ by } (296'). \qquad (300)$$

Now (263) is evidently true for $\nu - r - 1 \geq 0$, if $a_3 > 10$ and $r < \nu^{1/3}$. Hence, for $\nu \geq 2$,

$$\sum_{1 \leq \Sigma a_i \leq \nu^{\frac{1}{3}}} (-1)^{\Sigma a_i} \frac{(l_2 x)^{\nu - \Sigma a_i - 1}}{(\nu - \Sigma a_i - 1)!} \frac{1}{p_1^{a_1} \dots p_j^{a_j}}$$

$$= \frac{(l_2 x)^{\nu - 1}}{(\nu - 1)!} \sum_{1 \leq \Sigma a_i \leq \nu^{\frac{1}{3}}} (-1)^{\Sigma a_i} \frac{\omega^{\Sigma a_j}}{p_1^{a_1} \dots p_j^{a_j}} +$$

$$+ \theta \left\{ a_3 \frac{(l_2 x)^{\nu - 1}}{(\nu - 1)!} \omega . \frac{\nu^{2/3}}{\nu} \sum_{a_i \geq 0}^{\infty} 1/2^{a_1} \dots (j+1)^{a_j} \right\}$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{\Sigma a_i \geq 1} (-1)^{\Sigma a_i} \frac{\omega^{\Sigma a_i}}{p_1^{a_1} \dots p_j^{a_j}} +$$

$$+ \theta \left\{ \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \omega \sum_{\Sigma a_i \geq \nu^{\frac{1}{4}}} \frac{1}{p_1^{a_1} \dots p_j^{a_j}} \right\} +$$

$$+ \theta \left\{ a_{35} \frac{(l_2 x)^{\nu-2}}{(\nu-1)!} \nu^{3/4} \right\}, \quad \text{by (300)},$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{\Sigma a_i \geq 1} (-1)^{\Sigma a_i} \frac{\omega^{\Sigma a_i}}{p_1^{a_1} \dots p_j^{a_j}} + \theta \left\{ a^{36} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{\nu^{3/4}}{l_2 x} \right\}, \quad (301)$$

by virtue of (297).

Hence by (299) and (301) we get, for $l_2 x \geq \nu \geq 2$,

$$S_1 = f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} \sum_{\Sigma a_i \geq 1} (-1)^{\Sigma a_i} \times$$

$$\times \frac{\omega^{\Sigma a_i}}{p_1^{a_1} \dots p_j^{a_j}} + \theta \left(a_{37} \frac{\nu^{3/4}}{l_2 x} \right).$$

Also, by (238) for $l_2 x \geq \nu \geq 2$

$$\pi_\nu(x) = f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} \left\{ 1 + \theta \left(\frac{a_{11}}{l_2 x} \right) \right\}.$$

Hence

$$\pi_\nu(x) + S_1 = f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} \times$$

$$\times \sum_{a_i \geq 0} (-1)^{\Sigma a_i} \frac{\omega^{\Sigma a_i}}{p_1^{a_1} \dots p_j^{a_j}} + \theta \left(a_{38} \frac{\nu^{3/4}}{l_2 x} \right)$$

$$= \left\{ \prod_{p \mid p_1 \dots p_j} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} \times$$

$$\times \left\{ 1 + \theta \left[a_{38} \frac{\nu^{3/4}}{l_2 x} \prod_{p \leq 2^{\nu^{1/12}+1}} \left(1 + \frac{1}{p} \right) \right] \right\}$$

$$= \left\{ \prod_{p \mid p_1 \dots p_j} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} \times$$

$$\times \left\{ 1 + \theta \left(a_{39} \frac{\nu^{3/4+1/12}}{l_2 x} \right) \right\}. \quad (302)$$

Also by (298)

$$|S_2| \leqslant a_{29} \frac{x}{\log x} \frac{(l_2 x)^{\nu-2}}{(\nu-1)!}$$

$$\leqslant 10\, a_{29} \left\{ \prod_{p|p_1 \cdot\, p_j} \left(1+\frac{\omega}{p}\right)^{-1} \right\} \times$$

$$\times f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-2}}{(\nu-1)!} \prod_{p \leqslant 2^{\nu^{1/12}+1}} \left(1+\frac{1}{p}\right)$$

$$\leqslant a_{40} \left\{ \prod_{p|p_1 \ldots p_j} \left(1+\frac{\omega}{p}\right)^{-1} \right\} f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{\nu^{1/12}}{l_2 x}. \quad (303)$$

From (295), (302) and (303) the lemma follows on taking $a_{27} \geqslant a_{39} + a_{40}$.

LEMMA 58. *If* $1 \leqslant \nu \leqslant l_2 x$ *then*

$$\rho_\nu(x) = \left\{ 1 + \theta \left(a_{41} \frac{\nu^{12/13}}{l_2 x} \right) \right\} \times$$

$$\times f(\omega) \prod_p \left\{ 1 + \frac{\omega}{(p+\omega)(p-1)} \right\} \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!}. \quad (304)$$

In particular when $\nu = \nu_0 = [\log \log x]$

$$\rho_{\nu_0}(x) \sim \frac{1}{\sqrt{(2\pi)}} \frac{x}{\sqrt{(\log \log x)}}.$$

PROOF. By Lemma 55 we have,

$$\rho_\nu(x) = \pi_\nu(x) + \sum_{\substack{2 \leqslant m \\ t(m) \leqslant \nu}} \sum_{d|m} |\mu(d)|\, \pi'_{\nu - t(m)}(x/m^2 d, m)$$

$$= \pi_\nu(x) + \sum_{\substack{2 \leqslant m \leqslant \nu^{1/12}+1 \\ t(m) \leqslant \nu}} \sum_{d|m} |\mu(d)|\, \pi'_{\nu - t(m)}(x/m^2 d, m) +$$

$$+ \sum_{\substack{m > \nu^{1/12}+1 \\ t(m) \leqslant \nu}} \sum_{d|m} |\mu(d)|\, \pi'_{\nu - t(m)}(x/m^2 d, m)$$

$$= \pi_\nu(x) + \sum\nolimits_1 + \sum\nolimits_2, \text{ say.} \quad (305)$$

$$\left|\sum\nolimits_2\right| \leqslant \sum_{\substack{\sqrt{x/2} \geqslant m \geqslant \nu^{1/12}+1 \\ t(m) \leqslant \nu-1}} \sum_{\substack{d|m \\ d \leqslant x/2m^2}} |\mu(d)| \, \pi_{\nu-t(m)}(x/m^2 d) + \sum_{m \leqslant \sqrt{x}} \sum_{d|m} |\mu(d)|$$

$$\leqslant a_{42} x \frac{(l_2 x)^{\nu-2}}{(\nu-2)!} \sum_{\sqrt{x/2} \geqslant m \geqslant \nu^{1/12}} \frac{1}{m^2} \sum_{\substack{d|m \\ d \leqslant x/2m^2}} \frac{|\mu(d)|}{d \log x/m^2 d} + \sum_{m \leqslant \sqrt{x}} d(m),$$

by (73), noting that $\nu \leqslant l_2 x$,

$$\leqslant a_{43} x \frac{(l_2 x)^{\nu-2}}{(\nu-2)!} \left\{ \frac{4}{\log x} \sum_{m \geqslant \nu^{1/12}}^{\infty} \frac{\log m}{m^2} + \frac{1}{\log 2} \sum_{m \geqslant x^{1/4}}^{\infty} \frac{\log m}{m^2} \right\} +$$

$$+ a_{43} \sqrt{x} \log x,$$

$$\left[\text{since, } \sum_{d|m} \frac{|\mu(d)|}{d} = \prod_{p|m} \left(1+\frac{1}{p}\right) < \prod_{p \leqslant m} \left(1+\frac{1}{p}\right) \right],$$

$$\leqslant a_{44} \frac{x}{\log x} \frac{(l_2 x)^{\nu-2}}{(\nu-2)!} \frac{1}{\nu^{1/13}}$$

$$\leqslant a_{44} \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{\nu^{12/13}}{l_2 x}.$$

Hence by virtue of (259) we get if $l_2 x \geqslant \nu \geqslant 3$

$$\left|\sum\nolimits_2\right| \leqslant 10 \, a_{44} \frac{\nu^{12/13}}{l_2 x} f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!}. \tag{306}$$

Now for $2 \leqslant m \leqslant \nu^{1/12} + 1, l_2 x \geqslant \nu \geqslant 3$ we have

$$1 \leqslant t(m) \leqslant \left[\frac{\log m}{\log 2}\right] \leqslant \left[\frac{\log (\nu^{1/12}+1)}{\log 2}\right] \tag{307}$$

$$l_2 \, \overline{x/m^2 d} \geqslant \nu - t(m) \geqslant 2, \quad \text{if } d|m,$$

$$\prod_{p|m} p \leqslant m \leqslant \nu^{1/12} + 1 \leqslant 2^{(\nu - t(m))^{1/12}} + 1.$$

Hence by Lemma 57

$\pi'_{\nu-t(m)}(x/m^2 d, m)$

$$= \pi'_{\nu-t(m)}\left(x/m^2 d, \prod_{p|m}' m\right)$$

$$= \left\{ 1 + \theta \left(2\, a_{27} \frac{\nu^{5/6}}{l_2 x} \right) \right\} f \left(\frac{\nu - t(m)}{l_2\, (x/m^2 d)} \right) \times$$

$$\times \left\{ \prod_{p|m} \left(1 + \frac{\nu - t(m)}{p\, l_2\, (x/m^2 d)} \right)^{-1} \right\} \cdot \frac{x}{m^2 d} \frac{(l_2\, x/m^2 d)^{\nu - t(m) - 1}}{(\nu - t(m) - 1)!} \times$$

$$\times \frac{1}{\log\, (x/m^2 d)}$$

$$= \left\{ 1 + \theta \left(a_{45} \frac{\nu^{5/6}}{l_2\, x} \right) \right\} f(\omega) \frac{x}{\log x} \frac{1}{m^2 d} \frac{(l_2\, x)^{\nu - t(m) - 1}}{(\nu - t(m) - 1)!} \times$$

$$\times \prod_{p} \left\{ \left(1 + \frac{\nu - t(m)}{p\, l_2\, (x/m^2 d)} \right)^{-1} \right\} \tag{308}$$

as in (278)-(281). Now,

$$1 + \frac{\nu - t(m)}{p\, l_2 (x/m^2 d)} = 1 + \frac{\nu}{p\, l_2\, (x/m^2 d)} + \theta \left(4 \frac{\nu^{1/2}}{p\, l_2\, x} \right)$$

$$= 1 + \frac{\nu}{p\, l_2\, x} \left\{ 1 + \theta \left(\frac{8\, \nu}{\log x} \right) \right\} + \theta \left(4 \frac{\nu^{1/12}}{p\, l_2\, x} \right)$$

$$= \left(1 + \frac{\omega}{p} \right) \left\{ 1 + \theta \left(12 \frac{\nu^{1/12}}{l_2\, x} \right) \right\}.$$

Hence

$$\prod_{p|m} \left\{ 1 + \frac{\nu - t(m)}{p\, l_2\, x/m^2 d} \right\}^{-1} = \left\{ \prod_{p|m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} \left\{ 1 + \theta \left(30 \frac{\nu^{1/12}}{l_2\, x} \right) \right\}^{t(m)}$$

$$= \left\{ \prod_{p|m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} \left\{ 1 + \theta \left(a_{46} \frac{\nu^{1/6}}{l_2\, x} \right) \right\}, \tag{309}$$

by virtue of (307). Hence from (308) and (309) we get

$$\Pi'_{\nu - t(m)} (x/m^2 d, m)$$

$$= \left\{ 1 + \theta \left(a_{47} \frac{\nu^{5/6}}{l_2\, x} \right) \right\} f(\omega) \frac{x}{\log x} \times$$

$$\times \left\{ \prod_{p|m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} \frac{(l_2\, x)^{\nu - t(m) - 1}}{(\nu - t(m) - 1)!} \frac{1}{m^2 d}$$

$$= f(\omega) \frac{x}{\log x} \left\{ \prod_{p \mid m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} \frac{(l_2 x)^{\nu - t(m) - 1}}{(\nu - t(m) - 1)!} \frac{1}{m^2 d} +$$

$$+ \theta \left[a_{47} \frac{\nu^{5/6}}{l_2 x} f(\omega) \frac{x}{\log x} \left\{ \prod_{p \mid m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \frac{1}{m^2 d} \right]$$

$$= f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \frac{\omega^{t(m)}}{m^2 d} \left\{ \prod_{p \mid m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} +$$

$$+ \theta \left[a_{48} \frac{\nu^{5/6}}{l_2 x} f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \left\{ \prod_{p \mid m} \left(1 + \frac{\omega}{p} \right)^{-1} \right\} \frac{1}{m^2 d} \right]$$

by (263) and (307). (310)

From (305) and (310) we get, if $l_2 x \geqslant \nu \geqslant 3$,

$$\sum\nolimits_1 = f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \left\{ \sum_{2 \leqslant m \leqslant \nu^{1/12}+1} \frac{\omega^{t(m)}}{m^2} \prod_{p \mid m} \left(\frac{1 + 1/p}{1 + \omega/p} \right) + \right.$$

$$\left. + \theta \left[a_{48} \frac{\nu^{5/6}}{l_2 x} \sum_{m=2}^{\infty} \frac{1}{m^2} \prod_{p \mid m} \left(1 + \frac{1}{p} \right) \right] \right\}$$

$$= f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \left\{ \sum_{m=2}^{\infty} \frac{\omega^{t(m)}}{m^2} \prod_{p \mid m} \left(\frac{1 + 1/p}{1 + \omega/p} \right) + \right.$$

$$\left. + \theta \left(a_{49} \frac{\nu^{12/13}}{l_2 x} \right) \right\}, \qquad (311)$$

since

$$\sum_{m < \nu^{1/12}+1}^{\infty} \frac{\omega^{t(m)}}{m^2} \prod_{p} \left(1 + \frac{1}{p} \right) \leqslant a_{50}\, \omega \sum_{m \geqslant \nu^{1/12}}^{\infty} \frac{\log m}{m^2} .$$

By Lemma 50 and (311) we get, if $l_2 x \geqslant \nu \geqslant 3$,

$$\pi_\nu(x) + \sum\nolimits_1 = f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} \left\{ 1 + \sum_{m=2}^{\infty} \frac{\omega^{t(m)}}{m^2} \times \right.$$

$$\left. \times \prod_{p \mid m} \left\{ \frac{1 + 1/p}{1 + \omega/p} \right\} + \theta \left(a_{51} \frac{\nu^{12/13}}{l_2 x} \right) \right\}$$

$$= \left\{ 1 + \theta \left(a_{51} \frac{\nu^{12/13}}{l_2 x} \right) \right\} f(\omega) \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} +$$

$$+ \prod_{p} \left\{ 1 + \frac{\omega}{(p + \omega)(p - 1)} \right\}, \qquad (312)$$

by using equation (10), page 16, of Ingham's tract with

$$f(n) = \frac{\omega^{t(n)}}{n^2} \prod_{p \mid n} (1 + 1/p)/(1 + \omega/p).$$

From (305), (306) and (312) we get, if $l_2 x \geqslant \nu \geqslant 3$,

$$\rho_\nu(x) = \left\{ 1 + \theta \left(a_{53} \frac{\nu^{12/13}}{l_2 x} \right) \right\} \times$$
$$\times f(\omega) \prod_p \left\{ 1 + \frac{\omega}{(p+\omega)(p-1)} \right\} \frac{x}{\log x} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!}.$$

Hence (304) is truė for $l_2 x \geqslant \nu \geqslant 3$ if $a_{41} \geqslant a_{53}$. It is evidently true for $\nu = 1, 2$ with a suitable value of a_{41}. Hence (304) is true for $l_2 x \geqslant \nu \geqslant 1$.

The particular case $\nu = \nu_0$ follows immediately, since as $x \to \infty$

$$f(\nu_0/l_2 x) \to f(1) = 6/\pi^2;$$

$$\prod_p \left\{ 1 + \frac{\omega}{(p+\omega)(p-1)} \right\} \to \prod_p \left(1 - \frac{1}{p^2} \right)^{-1} = \pi^2/6,$$

$$\frac{(l_2 x)^{\nu_0 - 1}}{(\nu_0 - 1)!} \to \frac{1}{\sqrt{(2\pi)}} \frac{\log x}{\sqrt{(\log\log x)}},$$

by Stirling's theorem.

CHAPTER 8

Generalization of the results

In this section the following additional notations will be used:

$$1 \leqslant k < e, \quad \epsilon = \frac{1 - \log k}{2}, \quad \beta = (1/2)^{1/k}, \tag{313}$$

$d, d', d_1, d_2, \ldots$, are postive constants depending *at most* upon k and on no other parameter or variable.

$$d \geqslant d_1 + d_2 + 1, \quad c'' = \max\{c', d, d', d_1, \ldots, d_{36}, d_{37}\},$$

$$h''(\nu) = \prod_{m=1}^{\nu}\left(1+\frac{c''}{\sqrt{\nu}}\right), \text{ hence } h''(\nu) \geqslant h(\nu). \tag{314}$$

The most fundamental result of this section is:

If $k\,l_2 x \geqslant \nu \geqslant l_2 x$ then for all ν

$$|R_\nu(x)| \leqslant d\,\frac{x}{(\log x)^{1+\epsilon}}\,e^{\epsilon\nu}\,h''(\nu). \tag{315}$$

As in (18) and (19) we have,

$$|R_1(x)| \leqslant d_1\,\frac{x}{(\log x)^{1+\epsilon}}, \text{ when } k\,l_2 x \geqslant 1,$$

$$|R_2(x)| \leqslant d_2\,\frac{x}{(\log x)^{1+\epsilon}}, \text{ when } k\,l_2 x \geqslant 2.$$

Hence (315) is true when $\nu = 1, 2$.

We assume now that (315) is true for $1 \leqslant \nu \leqslant \nu'$, where ν' is a fixed integer $\geqslant 2$. (316)

From (316) we deduce as before that (315) is true for $\nu = \nu' + 1$ also. The method is precisely the same, there being only a few important changes. We use the notations $S_\nu(x, n)$, $E_\nu(x, n)$, etc. in the same sense as before.

It may be pointed out explicitly that if $l_2 x \leqslant \nu \leqslant k\,l_2\,x$ then as $x \to \infty$

$$\frac{(l_2\,x)^\nu/\nu!}{e^{\epsilon\nu}\,h''(\nu)/(\log x)^\epsilon}\ \Omega\ \frac{1}{\sqrt{\nu}}\,e^{\nu(1-\log k-\epsilon)}, \quad \text{by Stirling's theorem,}$$

$$\geqslant \frac{e^{\epsilon\,l_2 x}}{\sqrt{(k\,l_2\,x)}} \to \infty,$$

since $\epsilon > 0$ by virtue of (313). Hence $\psi(\nu, x)$ dominates $R_\nu(x)$.

Lemma 59. *If* $1 \leqslant \nu \leqslant \nu'$, $k\,l_2 x \geqslant \nu \geqslant l_2 x$ *and* (316) *holds, then*

$$|E_\nu(x)| \leqslant d_3\,e^{\epsilon\nu}\,h''(\nu)/(\log x)^\epsilon. \tag{317}$$

Proof. $\int_{a_1}^{\infty} \frac{|R_\nu(u)|}{u^2}\,du$ is convergent by virtue of Lemma 12;

(23) being now a complètely proved result.

Also in equation (81),

$$\frac{\pi_\nu(x)}{x} \leqslant c_{14} \frac{(l_2 x + c_{15})^{\nu-1}}{(\nu-1)!} \frac{1}{\log x}, \qquad \text{by (73)},$$

$$\leqslant c_{14} e^{c_{15}}. \tag{318}$$

$$\int_x^\infty \frac{|R_\nu(u)|}{u^2} du = \int_x^{e^{e^\nu}} \frac{|R_\nu(u)|}{u^2} du + \int_{e^{e^\nu}}^\infty \frac{|R_\nu(u)|}{u^2} du$$

$$\leqslant \int_x^{e^{e^\nu}} \frac{|R_\nu(u)|}{u^2} du + d_4 \frac{\nu^\nu}{\nu!} \frac{\sqrt{\nu}\, h(\nu)}{e^\nu}$$

on putting $l_2 x = \nu$ in (82),

$$\leqslant \int_x^{e^{e^\nu}} \frac{|R_\nu(u)|}{u^2} du + d_4 h''(\nu), \tag{319}$$

since $h''(\nu) \geqslant h(\nu)$ and $\frac{\nu^\nu}{\nu!} \frac{\sqrt{\nu}}{e^\nu} \leqslant \frac{1}{\sqrt{(2\pi)}}$.

Also by (316),

$$\int_x^{e^{e^\nu}} \frac{|R_2(u)|}{u^2} du \leqslant d\, e^{\epsilon\nu} h''(\nu) \int_x^{e^{e^\nu}} \frac{du}{u(\log u)^{1+\epsilon}},$$

since $k\, l_2 u \geqslant k\, l_2 x \geqslant \nu \geqslant l_2 e^{e^\nu} \geqslant l_2 u$,

$$\leqslant d\, e^{\epsilon\nu} h''(\nu) \int_x^\infty \frac{du}{u(\log u)^{1+\epsilon}}$$

$$= \frac{d}{\epsilon} \frac{e^{\epsilon\nu}}{(\log x)^\epsilon} h''(\nu). \tag{320}$$

Hence from (319) and (320),

$$\int_x^\infty \frac{|R_\nu(u)|}{u^2} du \leqslant \left(\frac{d}{\epsilon} + d_4\right) \frac{e^{\epsilon\nu}}{(\log x)^\epsilon} h''(\nu),$$

since $e^\nu \geqslant \log x$ when $\nu \geqslant l_2 x$.

From this and from (318), the result follows by virtue of equation (81).

LEMMA 60. *If* (316) *holds,* $1 \leqslant \nu \leqslant \nu'$, $1 \leqslant n \leqslant 6\,(k\,l_2 x - \nu \log 2)$, *then, when* $k\,l_2 x \geqslant \nu \geqslant l_2 x$

$$| E_\nu (x, n) | \leqslant d' (\log x)^{n-\epsilon} e^{\epsilon\nu} \sqrt{\nu}\, h'' (\nu), \tag{321}$$

where

$$d' \geqslant b + d_3, \tag{322}$$

b being the constant of equation (86) *and* d_3, *of Lemma* 59.

PROOF. By induction through ν as in (88).

(*A*) (321) *is true when* $\nu = 1$.

PROOF: When $k\,l_2 x \geqslant 1, x \geqslant e$; hence for any $n \geqslant 1$ we have by Lemma 14,

$$| E_1 (x, n) | \leqslant b_3 (\log x)^{n-1} \leqslant d' (\log x)^{n-1}$$
$$\leqslant d' (\log x)^{n-\epsilon}.$$

(*B*) *If* $1 \leqslant n \leqslant 30\, l_2 x$, $l_2 x \leqslant \nu \leqslant k\, l_2 x$, $\nu \geqslant 2$, *then*

$$\begin{aligned}\nu S_\nu (x, n) = \sum_{l=0}^{n} \Big\{ & \sum_{p \leqslant x^{1-\beta}} \frac{(\log p)^l}{p} S_{\nu-1} (x/p, n-l) + \\ & + \sum_{q_{\nu-1} \leqslant x^\beta} \frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}} S (x/q_{\nu-1}, l) - \\ & - S (x^{1-\beta}, l)\; S_{\nu-1} (x^\beta, n-l) \Big\} + \\ & + \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t} (x/p^t, n) + \\ & \qquad + \theta \left\{ d_5 (\log x)^{n-\epsilon} e^{\nu t} h'' (\nu) \right\}\end{aligned}$$

PROOF. As before. We have only to note that

$$\sum_{t=2}^{\nu} \frac{(l_2 x + b_8)^{\nu-t}}{(\nu - t)!} \sum_p \frac{\log p}{p^t} \leqslant d_6 \log x \sum_{t=2}^{\infty} \sum_p \frac{\log p}{p^t}.$$

Hence in equation (100)

$$\Big| \sum \Big| \leqslant d_7 (\log x)^n l_2 x\, \nu^2$$
$$\leqslant d_7 (\log x)^{n-\epsilon} e^{\epsilon\nu} \nu^3$$
$$\leqslant d_8 (\log x)^{n-\epsilon} e^{\epsilon\nu} h'' (\nu).$$

(C) If $2 \leqslant \nu \leqslant \nu'$; $1 \leqslant n \leqslant 6\,(k\,l_2 x - \nu \log 2)$ *and* (321) *is true for the index* $(\nu - 1)$, *then when* $k\,l_2 x \geqslant \nu \geqslant l_2 x$ (323)

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-\beta}} \frac{(\log p)^l}{p} S_{\nu-1}(x/p, n-l)$$

$$= \sum_{l=0}^{n-1} \binom{n}{l} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} \frac{(-1)^{t-1}}{(n-l)^t}\, T(\nu - r - t - 1, l, x^{1-\beta}) \; +$$

$$+ \sum_{r=0}^{\nu-1} B_r\, T(\nu - r - 1, n, x^{1-\beta}) \; +$$

$$+ \theta \left\{ d' (\log x)^{n-\epsilon} e^{\nu\epsilon} h''(\nu - 1)\, \nu^{3/2} \left(1 + \frac{d_9}{\nu}\right) \right\}, \quad (324)$$

where $T(\nu, l, x^{1-\beta})$ *is used in the same sense as in Lemma* 20.

Proof. As in Lemma 19 we have only to consider the value of the expression

$$\left| \sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-\beta}} \frac{\log p^l}{p} E_{\nu-1}(x/p, n-l) \right|$$

under the given conditions.

Now by virtue of (323) and (321) we have, if $u \leqslant x^{1-\beta}$

$$| E_{\nu-1}(x/u, n-l) | \leqslant d' (\log \overline{x/u})^{n-l-\epsilon} e^{(\nu-1)\epsilon} \sqrt{(\nu-1)}\, h''(\nu-1)$$

$$\leqslant d' (\log \overline{x/u})^{n-l-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu-1)$$

if $1 \leqslant n - l \leqslant 6\,\{k\,l_2\,\overline{x/u} - (\nu - 1)\log 2\}$; $k\,l_2\,\overline{x/u} \geqslant \nu - 1 \geqslant l_2\,\overline{x/u}$, a fortiori, if

$$\left.\begin{aligned} 1 \leqslant n \leqslant 6\,(k\,l_2 x - \nu \log 2);\; 0 \leqslant l \leqslant n - 1; \\ k\,l_2 x \geqslant \nu \geqslant l_2\,\overline{x/u} + 1 \end{aligned}\right\} \quad (325)$$

since $k\,l_2\,\overline{x/u} \geqslant k\,l_2 x^{\beta} = k\,l_2 x - k \log \beta = k\,l_2 x - \log 2.$

Also by (88), if $\nu \geqslant 2$ then we have for all values of x for which $l_2\overline{x/u} \geqslant \nu - 1$, a fortiori, for those values of x for which $l_2 x - 1 \leqslant \nu - 1 \leqslant l_2\overline{x/u}$,

$$|E_{\nu-1}(x/u, n-l)| \leqslant b\,(\log \overline{x/u})^{n-l-1}\,\frac{(l_2\overline{x/u})^{\nu}}{\nu!}\,\sqrt{\nu}\,h(\nu-1)$$

$$\leqslant b\,(\log \overline{x/u})^{n-l}\,\sqrt{\nu}\,h''(\nu-1)$$

$$\leqslant d'\,(\log \overline{x/u})^{n-l-\epsilon}\,e^{\nu\epsilon}\,\sqrt{\nu}\,h''(\nu-1),$$

$$\left\{\text{since } \frac{e^{\nu}}{\log}\,\overline{x/u} \geqslant \frac{e^{\nu}}{\log x} \geqslant 1\right\},$$

if $1 \leqslant n - l \leqslant 30\,l_2 x/u$, a fortiori, if $1 \leqslant n \leqslant 6\,(k\,l_2 x - \nu \log 2)$

$$0 \leqslant l \leqslant n - l. \tag{325'}$$

Since $l_2\,x \geqslant 2/k$ by hypothesis and for $u \leqslant x^{1-\beta}$

$$30\,l_2\overline{x/u} \geqslant 30\,l_2 x + 30 \log \beta \geqslant 18\,l_2 x + \left(\frac{24}{k} - 30\,\frac{\log 2}{k}\right)$$

$$\geqslant 18\,l_2 x \geqslant 6\,(k\,l_2 x - \nu \log 2). \tag{325''}$$

From (325) and (325′) we get

$$|E_{\nu-1}(x/u, n-l)| \leqslant d'\,(\log \overline{x/u})^{n-l-\epsilon}\,e^{\nu\epsilon}\,\sqrt{\nu}\,h''(\nu-1), \tag{326}$$

if $1 \leqslant n \leqslant 6\,(k\,l_2 x - \nu \log 2),\ 0 \leqslant l \leqslant n - l,\ k\,l_2\,x \geqslant \nu \geqslant l_2 x,\ u \leqslant x^{1-\beta}$.

Also, if $k\,l_2\overline{x/u} \geqslant \nu - 1 \geqslant k\overline{x/u},\ u \leqslant x^{1-\beta}$ we get by Lemma 59,

$$|E_{\nu-1}(x/u, 0)| \leqslant d_3\,e^{\epsilon\nu}\,h''(\nu-1)/(\log \overline{x/u})^{\epsilon}$$

$$\leqslant d'\,e^{\epsilon\nu}\,h''(\nu-1)/(\log \overline{x/u})^{\epsilon}.$$

A fortiori, this is true if $k\,l_2\,x \geqslant \nu \geqslant l_2\,\overline{x/u} + 1$. (326′)

By (37) we have for *any* $\nu \geqslant 2$

$$|E_{\nu-1}(x/u, 0)| \leqslant c_5\,\frac{(l_2\overline{x/u})^{\nu-1}}{(\nu-1)!}\,\frac{\sqrt{(\nu-1)}\,h(\nu-1)}{(\log \overline{x/u})}$$

$$\leqslant b\,\sqrt{(\nu-1)}\,h(\nu-1) < d'\,\sqrt{\nu}\,h''(\nu-1),$$

for all values of x for which $l_2\,\overline{x/u} \geqslant \nu - 1$; the restriction $\nu \leqslant \nu'$ being removed by the fact that (23) has been shown to be true.

Hence it follows that, in particular, for those values of x which satisfy $l_2\overline{x/u} + 1 \geqslant \nu \geqslant l_2 x$.

$$| E_{\nu-1}(x/u, 0) | \leqslant d' \sqrt{\nu}\, h''(\nu-1) \leqslant d' (\log \overline{x/u})^{-\epsilon} e^{\epsilon\nu} \sqrt{\nu}\, h''(\nu-1). \tag{326''}$$

Hence from (326), (326′) and (326″) we get that, if $u \leqslant x^{1-\beta}$ then under the conditions of (324)

$$| E_{\nu-1}(x/u, n-l) | \leqslant d' (\log \overline{x/u})^{n-l-\epsilon} e^{\epsilon\nu} \sqrt{\nu}\, h''(\nu-1), \tag{327}$$

for $0 \leqslant l \leqslant n$.

From (327) we get

$$\left| \sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-\beta}} \frac{(\log p)^l}{p} E_{\nu-1}(x/p, n-l) \right|$$

$$\leqslant d' e^{\epsilon\nu} \sqrt{\nu}\, h''(\nu-1) \sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-\beta}} \frac{(\log p)^l}{p} \frac{(\log \overline{x/p})^{n-l}}{(\log \overline{x/p})^{\epsilon}},$$

$$= d' e^{\epsilon\nu} \sqrt{\nu}\, h''(\nu-1) (\log x)^{n-\epsilon} \sum_{p \leqslant x^{1-\beta}} \frac{1}{p(1 - lp/lx)^{\epsilon}}$$

$$\leqslant d' e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu-1) (\log x)^{n-\epsilon} \sum_{p \leqslant x^{1-\beta}} \frac{1}{p(1 - lp/lx)}$$

$$\leqslant d' e^{\epsilon\nu} \sqrt{\nu}\, h''(\nu-1) (\log x)^{n-\epsilon} \left\{ \sum_{p \leqslant x^{1-\beta}} \left(\frac{1}{p} + \frac{1}{\beta} \frac{lp}{plx} \right) \right\}$$

$$\leqslant d' e^{\epsilon\nu} \sqrt{\nu}\, h''(\nu-1) (\log x)^{n-\epsilon} \{ l_2 x + d_{10} \}$$

$$\leqslant d' e^{\epsilon\nu} h''(\nu-1) (\log x)^{n-\epsilon} \nu^{3/2} \left(1 + \frac{d_{10}}{\nu}\right), \quad \text{since } \nu \geqslant l_2 x.$$

(*D*) *If* $k\,l_2 x \geqslant \nu \geqslant l_2 x$, $1 \leqslant n \leqslant 30\, l_2 x$ *then Lemma* 23 *holds for any* $\nu \geqslant 2$ *with* β *in place of* α *if we replace the error term by*

$$\theta \{ d_{11} (\log x)^{n-\epsilon} e^{\nu\epsilon} h''(\nu-1) \}.$$

PROOF. We see, in the first instance, that if

$$k\,l_2 x \geqslant \nu \geqslant 1,\ 1 \leqslant n \leqslant 30\, l_2 x, t \geqslant 1$$

$$\sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^t}\, T(\nu-1, l, x^{1-\beta})$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \frac{(\log x)^n}{(n-l)^t} I_{\nu-1,l,\beta} +$$

$$+\theta \left\{ d_{12} \frac{(\log x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2 \right\}. \qquad (328)$$

This follows as in Lemma 20, noting that

$$\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \leqslant 2\, \frac{(l_2 x)^{\nu+1}}{(\nu+1)!}\, \frac{\nu^2}{(l_2 x)^2} \leqslant 2\, k^2\, \frac{(l_2 x)^{\nu+1}}{(\nu+1)!}. \qquad (329)$$

From (328) it follows that (127) holds for $\nu \geqslant 2$, $a = \beta$ and for those values of x for which $k\, l_2 x \geqslant \nu$ provided we replace the error term by $d_{13} (\log x)^n \nu^3$, since for $k\, l_2\, x \geqslant \nu$,

$$\sum_{r=0}^{\nu-2} |B_r| \sum_{t=1}^{\nu-r-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \leqslant (\log x)\, \nu \sum_{r=0}^{\infty} |B_r|. \qquad (330)$$

Similarly, Lemma 22 is true for $k\, l_2\, x \geqslant \nu \geqslant 2$, $1 \leqslant n \leqslant 30\, l_2\, x$ if we replace the error term by $d_{14} (\log x)^n \nu^2$.

Hence if $\nu \geqslant 2$, Lemma 23 is true for all values of x for which $k\, l_2 x \geqslant \nu$, a fortiori, for those values of x for which $k\, l_2\, x \geqslant \nu \geqslant l_2\, x$ if we replace the error term by

$$(d_{13} + d_{14}) (\log x)^n \nu^3 \leqslant d_{15} (\log x)^n\, h''(\nu - 1).$$

From this the result follows immediately, since $e^\nu / \log x \geqslant 1$ if $\nu \geqslant l_2\, x$.

(*E*) *Lemmas* 24 *and* 25 *are true under the conditions of* (323) *if we replace the error terms by* $(d_{16}, d_{17})\ (\log x)^{n-\epsilon}\, e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu - 1)$ *and write* β *in place of* a.

PROOF. As before. We have only to make use of the equations (325) and (330).

(*F*) *Under the conditions of* (323), *Lemma* 26 *is true with* $\alpha = \beta$ *if we replace the error term by*

$$d'(\log x)^{n-\epsilon} e^{\nu\epsilon} h''(\nu-1)\, \nu^{3/2}(1 + d_{18}/\nu).$$

PROOF. (*C*) – (*E*).

(*G*) *Under the conditions of* (323), *Lemma* 27 *holds with* β *in place of* α, *provided the error term is replaced by*

$$d_{19}(\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu-1).$$

PROOF. In equation (162), we have if $s'' = [6(k\, l_2 x - \nu \log 2)]$,

$$\left|\sum_{q_{\nu-1} \leqslant x^\beta} \frac{(\log q^{\nu-1})^{s''}}{q_{\nu-1}}\right| \leqslant d_{20} \frac{(l_2 x + b_8)^{\nu-1}}{(\nu-1)\,!} \nu (\log x)^{s''} \beta^{s''}$$

$$\leqslant d_{21}(\log x)^{s''+1} \nu\, \beta^{s''}.$$

Also,

$$\beta^{s''} \leqslant \frac{2}{e^{(\log 2/k)\, 6\, (kl_2 x - k \log 2\, l_2 x)}} = \frac{2}{e^{6 \log 2\, (1 - \log 2)\, l_2 x}} \leqslant \frac{2}{\log x}, \tag{331}$$

as

$$6 \log 2\, (1 - \log 2) > 1.$$

Hence,

$$\left|\sum_{q_{\nu-1} \leqslant x^\beta} \frac{(\log q_{\nu-1})^{s''}}{q_{\nu-1}}\right| \leqslant d_{22}(\log x)^{s''} \nu$$

$$\leqslant d_{23}(\log x)^{s''-\epsilon} e^{\nu\epsilon} h''(\nu-1).$$

The rest follows as before, by using equations (330) and (331).

(*H*) *Under the conditions of* (323),

$\nu S_\nu(x, n)$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!}\left(-\frac{t+1}{n^{t+1}}\right) +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-3} B_r \sum_{t=2}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} +$$

$$+ \sum_{t'=1}^{t-1} \frac{1}{n^{t-t'}} S(t'+1) +$$

$$+ (\log x)^n (l_2 x + B) \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \frac{1}{n^t} +$$

$$+ \frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)!} +$$

$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n) +$$

$$+ \theta \left\{ d'(\log x)^{n-\epsilon} e^{\epsilon\nu} h''(\nu+1)\, \nu^{3/2} \left(1 + \frac{d_{24}}{\nu}\right) \right\}.$$

PROOF. It is easily seen by using (329) that Lemma 29 is true for $kl_2x \geqslant \nu \geqslant 1$, $1 \leqslant n \leqslant 30 l_2 x$, $\alpha = \beta$, if we replace the error term by

$$\theta \left\{ d_{25} \frac{(l_2\, x)^{\nu+2}}{(\nu+2)\,!} (\log x)^{n-1} \nu^3 \right\}.$$

Hence by (330), if $\nu \geqslant 2$, Lemma 30 is true for all values of x for which $k\, l_2 x \geqslant \nu$, provided we replace the error term by

$$d_{26} (\log x)^n \nu^4 \leqslant d_{27} (\log x)^n h'' (\nu - 1).$$

Hence if $k\, l_2 x \geqslant \nu \geqslant l_2 x$, the modulus of the error term in Lemma 30

$$\leqslant d_{27} (\log x)^n h'' (\nu-1) \leqslant d_{27} (\log x)^{n-\epsilon} e^{\nu\epsilon} h'' (\nu - 1).$$

Also using (326), we see that Lemma 28 is true with β for α if we replace the error term by

$$\theta \left\{ d_{28} (\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h'' (\nu-1) \right\}.$$

From these considerations the result follows immediately from (B), (F), (G) and Lemma 32.

(I) If $\nu' \geqslant \nu \geqslant 3$, $1 \leqslant n \leqslant 6\, (k\, l_2 x - \nu \log 2)$ *and* (321) *is true for the indices* $1, 2, \ldots, \nu - 2$, *then, when* $k\, l_2 x \geqslant \nu \geqslant l_2 x$,

$$\sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} \cdot S_{\nu-t} \left(\frac{x}{p^t}, n\right)$$

$$= (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t-1} B^r \times$$

$$\times \sum_{t'=1}^{\nu-r-t} (-1)^{t'-1} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}} +$$

$$+ \theta \left\{ d_{29} (\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu-1) \right\}.$$

PROOF. As in (204) we split up the given sum in $\sum_1, \sum_2, \sum_3$.

For $2 \leqslant t \leqslant \nu - 1, k\, l_2 x \geqslant \nu \geqslant l_2 x, \nu \geqslant 3$,

$$k\, l_2 \left\{ x/(\log x)^t \right\} \geqslant \nu - t, \tag{332}$$

if $x \geqslant d_{30}$. For (332) is true if $t/k \geqslant t\, l_2 x/(\log x - t\, l_2 x)$ or if $1/k \geqslant l_2 x/\log x - k\,(l_2 x)^2$.

Again,

$$l_2 \left\{ x/(\log x)^{\nu-1} \right\} \geqslant \log \log x^\beta = l_2 x - \frac{\log 2}{k}, \tag{333}$$

if $x \geqslant d_{31}$, since (333) is true if,

$$(\log x)^{kl_2 x - 1} \leqslant x^{1-\beta}.$$

Now the result (I) is evidently true for any finite value of x, in particular, for $x \leqslant d_{30} + d_{31}$, $kl_2 x \geqslant \nu \geqslant l_2\, x, \nu \geqslant 3$, $1 \leqslant n \leqslant 6$ $(kl_2 x - \nu \log 2)$ if the constant d_{29} is suitably chosen. Hence we may assume without loss of generality that $x > d_{30} + d_{31}$, so that both (332) and (333) are true.

It follows from the hypothesis that for

$$p \leqslant \log x,\ 2 \leqslant t \leqslant \nu - 1,\ k\, l_2 \overline{x/p^t} \geqslant \nu - t \geqslant l_2 \overline{x/p^t},$$
$$1 \leqslant n \leqslant 6(k\, l_2 \overline{x/p^t} - \overline{\nu - t} \log 2)$$

$$\begin{aligned} | E_{\nu-t}(x/p^t, n) | &\leqslant d' (\log \overline{x/p^t})^{n-\epsilon} e^{\epsilon(\nu-t)} \sqrt{\nu - t}\, h''(\nu-1) \\ &\leqslant d' (\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu-1). \end{aligned} \tag{334}$$

A fortiori, by virtue of (332) and (333), equation (334) is true if

$$k\, l_2 x \geqslant \nu \geqslant k\, l_2 \overline{x/p^t} + t,$$

$$1 \leqslant n \leqslant 6(k\, l_2 x - \nu \log 2).$$

Also by (88), if $2 \leqslant t \leqslant \nu - 1$, $\nu \geqslant 3$, $p \leqslant \log x$, then

$$\begin{aligned} |E_{\nu-t}(x/p^t, n)| &\leqslant b(\log x/p^t)^{n-1} \frac{(l_2\, \overline{x/p^t})^{\nu-t+1}}{(\nu-t+1)!} \sqrt{\nu - t}\, h(\nu - t) \\ &\leqslant d'(\log x)^n \sqrt{\nu}\, h''(\nu - 1) \\ &\leqslant d'(\log x)^{n-\epsilon} e^{\nu\epsilon} h''(\nu - 1) \sqrt{\nu}, \end{aligned} \tag{335}$$

if $l_2\ x \leqslant \nu \leqslant l_2\, \overline{x/p}^t + t$; $1 \leqslant n \leqslant l_2\, \overline{x/p}^t$. A fortiori, if

$$l_2 x \leqslant \nu \leqslant l_2\, \overline{x/p}^t + t;\ 1 \leqslant n \leqslant 6\ (k\, l_2 x - \nu \log 2)$$

by virtue of (333) and (325″).

From (334) and (335) we get

$$|E_{\nu-t}(x/p^t, n)| \leqslant d'(\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu - 1), \tag{336}$$

if $p \leqslant \log x$ and under the given conditions.

Hence,

$$\begin{aligned} \Big|\sum\nolimits_2\Big| &\leqslant d'(\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu - 1) \sum_{t=2}^{\infty} \sum_{p} 1/p^t \\ &= d_{32}(\log x)^{n-\epsilon} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu - 1). \end{aligned} \tag{337}$$

$$\begin{aligned} \Big|\sum\nolimits_3\Big| &\leqslant b_7(\log x)^n \nu \sum_{t=2}^{\nu-1} \sum_{x/2 \geqslant p^t \geqslant (\log x)^t} \frac{1}{p^t} \frac{(l_2\, \overline{x/p^t} + b_8)^{\nu-t}}{(\nu-t)!} \\ &\leqslant e^{b_8} b_7 (\log x)^{n+1} \nu \sum_{p > \log x}^{\infty} \sum_{t=2}^{\infty} 1/p^t \\ &\leqslant d_{33}(\log x)^{n-\epsilon} e^{\nu\epsilon} h''(\nu - 1). \end{aligned} \tag{338}$$

Treating $\sum_1$ as before and replacing $l_2 x$ by ν in the error term the result follows from (337) and (338).

(*J*) (321) *is true.*

PROOF. We now proceed precisely as before and the result follows immediately since we have only to simplify the expression in the dominant term.

LEMMA 61. *(315) holds for all ν. Hence, if $\nu \geqslant 1$, then when $k l_2 x \geqslant \nu \geqslant l_2 x$,*

$$\pi_\nu(x) = \psi(\nu, x) + \theta \left\{ d . \frac{x\, e^{\nu\epsilon}}{(\log x)^{1+\epsilon}} h''(\nu) \right\}. \tag{339}$$

PROOF. Lemma 60 gives in detail the method of extending the results of Chapter 4 to the case $k l_2 x \geqslant \nu \geqslant l_2 x$. In precisely the same way we get:

(A) *If* (316) *holds,* $2 \leqslant \nu \leqslant \nu'$, *then when* $k l_2 x \geqslant \nu + 1 \geqslant l_2 x$

$$\sum_{q_\nu \leqslant x^\beta} \pi(x/q_\nu) = \frac{x}{\log x} \sum_{r=0}^{\nu} B_r \frac{(l_2 x^\beta)^{\nu - r}}{(\nu - r)!} + \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \times$$

$$\times \sum_{t=1}^{\nu - r} (-1)^{t-1} \frac{(l_2 x^\beta)^{\nu - r - t}}{(\nu - r - t)!} \sum_{n=1}^{\infty} \frac{\beta^n}{n^t} +$$

$$+ \theta \left\{ d_{34} \frac{x}{(\log x)^{1+\epsilon}} e^{\nu\epsilon} \sqrt{\nu}\, h''(\nu) \right\}.$$

REMARK. In equation (216) we have $m = [6(1 - \log 2) k_2 x - 2]$ and use equation (331).

(B) *If* (316) *holds,* $2 \leqslant \nu \leqslant \nu'$, *then when* $k l_2 x \geqslant \nu + 1 \geqslant l_2 x$, *equation* (223) *holds with* β *for* α *if the error term is replaced by*

$$\theta \left\{ d \frac{x\, e^{(\nu+1)\epsilon}}{(\log x)^{1+\epsilon}} \nu\, h''(\nu) \left(1 + \frac{d_{35}}{\nu} \right) \right\}.$$

(C) *Lemma* 44 *holds if* $2 \leqslant \nu \leqslant \nu'$, (316) *holds and* $k l_2 x \geqslant \nu + 1 \geqslant l_2 x$ *provided the error term is replaced by*

$$\theta \left\{ d \frac{x\, e^{(\nu+1)\epsilon}}{(\log x)^{1+\epsilon}} \nu\, h''(\nu) \left(1 + \frac{d_{36}}{\sqrt{\nu}} \right) \right\}.$$

(D) *Lemma* 45 *holds if* $2 \leqslant \nu \leqslant \nu'$, (316) *holds and* $k l_2 x \geqslant \nu + 1 \geqslant l_2 x$ *provided the error term is replaced by*

$$d_{37} \frac{x\, e^{\nu\epsilon} h''(\nu)}{(\log x)^{1+\epsilon}}.$$

REMARK. We proceed as before and use the same reasoning as in (H) of Lemma 60.

(*E*) *If* $2 \leqslant \nu \leqslant \nu'$, (316) *holds, then, when* $k l_2 x \geqslant \nu + 1 \geqslant l_2 x$,
$(\nu+1)\, \pi_{\nu+1}(x)$

$$= \frac{x}{\log x} \sum_{r=0}^{\nu} (\nu + 1 - r)\, B_r \frac{(l_2 x)^{\nu - r}}{(\nu - r)\,!} + B \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)\,!} +$$

$$+ \frac{x}{\log x} \sum_{t=2}^{\nu} (-1)^{t-1} A(t) \sum_{r=0}^{\nu - t} B_r \frac{(l_2 x)^{\nu - r - t}}{(\nu - r - t)\,!} +$$

$$+ \theta \left\{ d \frac{x\, e^{(\nu+1)\epsilon}}{(\log x)^{1+\epsilon}}\, \nu\, h''(\nu + 1) \right\}.$$

PROOF. Follows from Lemma 43 (*C*), (*D*) and equation (314).

(*F*) (316) *is true for all* ν.

PROOF. Follows immediately from (*E*) as before.

LEMMA 62. *If* $l_2 x \leqslant \nu \leqslant k\, l_2 x$; $\nu \geqslant 1$, *then*

$$\psi(\nu, x) = \left\{ 1 + \theta \left(\frac{d_{38}}{l_2 x} \right) \right\} f(\omega) \frac{(l_2 x)^{\nu - 1}}{(\nu - 1)\,!} \frac{x}{\log x}, \tag{340}$$

where $\omega = \nu / l_2 x$ *and* $f(z)$ *has the same meaning as in Lemma* 1.

PROOF. As before. We have only to note that

$$f(z) > 0 \text{ for } z \geqslant 0 \tag{341}$$

and, in equation (265)

$$\left|\sum\nolimits_2\right| < \sum_{r=r'+1}^{\infty} |B_r| \frac{(l_2 x)^{\nu - r - 1}}{(\nu - r - 1)!}$$

$$\leqslant \frac{(l_2 x)^{\nu - 2}}{(\nu - 2)!} \sum_{r=r'+1}^{\infty} |B_r|\, e^r$$

$$\leqslant d_{40} \frac{(l_2 x)^{\nu-2}}{(\nu - 2)!} \sum_{r=r'+1}^{\infty} \frac{1}{2^r}, \quad \text{by Lemma 2,}$$

$$\leqslant d_{41} \frac{(l_2 x)^{\nu - 2}}{(\nu - 1)!}.$$

LEMMA 63. *If* $l_2 x \leqslant \nu \leqslant k\, l_2 x, \nu \geqslant 1$, *then*

$$\pi_\nu(x) = \left\{ 1 + \theta \left(\frac{d_{42}}{l_2 x} \right) \right\} f(\omega) \frac{(l_2 x)^{\nu - 1}}{(\nu - 1)!} \frac{x}{\log x}.$$

PROOF.

$$\frac{e^{\nu\epsilon} h^{\nu}(\nu)}{(\log x)^{\epsilon} \dfrac{(l_2 x)^{\nu-1}}{(\nu-1)!}} \leqslant \frac{d_{43}}{l_2 x} \leqslant \frac{d_{44}}{l_2 x} f(\omega)$$

by (341) and Stirling's theorem. Hence the lemma follows from (339) and (340).

LEMMA 64. *If* $1 \leqslant \nu \leqslant k\, l_2 x, k < e$, *then*

$$\pi_\nu(x) = \left\{ 1 + \theta \left(\frac{d_{45}}{l_2 x} \right) \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}, \tag{342}$$

where d_{45} *depends only on* k.

In particular, equation (5) *holds.*

PROOF. (342) is a combined statement of Lemmas 50 and 63. If now $\nu \sim k\, l_2 x, k < e$, then when $x \geqslant x_0 = x_0(k)$,

$$\nu \leqslant \tfrac{1}{2}(k+e)\, l_2 x, \text{ also } \frac{k+e}{2} < e.$$

Hence, when $x \geqslant x_0$ we get by (342)

$$\pi_\nu(x) = \left\{ 1 + \theta \left(\frac{d_{46}}{l_2 x} \right) \right\} f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x},$$

where d_{46} depends only upon k.

Hence as $x \to \infty$

$$\pi_\nu(x) \sim f(\omega) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x} \sim f(k) \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x},$$

since, when $x \geqslant d_{47}$, $|\omega - k| < 1$ and hence by (260)

$$|f(\omega) - f(k)| \leqslant a_3 |\omega - k|.$$

This tends to zero as $x \to \infty$ by hypothesis. Hence the lemma.

This completes the investigation of $\pi_\nu(x)$. By trivial modification of Chapter 6 we prove

LEMMA 65. *If* $1 \leqslant \nu \leqslant k\, l_2 x, k < 2$ *then*

$$\sigma_\nu(x) \sim f(\omega) \left\{ \prod_p \left(1 - \frac{\omega^2}{p^2}\right)^{-1} \right\} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}, \; as\ x \to \infty, \tag{343}$$

and

$$\rho_\nu(x) \sim f(\omega) \prod_p \left\{ 1 + \frac{\omega}{(p+\omega)(p-1)} \right\} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}, \; as\ x \to \infty. \tag{344}$$

Hence as before, equations (6) and (7) hold as particular cases of (343) and (344) respectively.

It may be remarked that the restriction $k < 2$ in the case of $\sigma_\nu(x)$ and $\rho_\nu(x)$ comes from the convergence of the series $\sum_m \omega^{F(m^2)}/m^2$ in the case of $\sigma_\nu(x)$ and $\sum_{a=0}^{\infty} (\omega/2)^a$ in the case of $\rho_\nu(x)$ We shall see later that in the case of $\sigma_\nu(x)$ this restriction is a genuine one.

CHAPTER 9

DIRECT PROOFS FOR $\sigma_\nu(x)$ AND $\rho_\nu(x)$

In the last two chapters we deduced the asymptotic formulae for $\sigma_\nu(x)$ and $\rho_\nu(x)$ from the corresponding result for $\pi_\nu(x)$. Instead, we may start from the recurrence relations for $\sigma_\nu(x)$ and $\rho_\nu(x)$ and proceed as before. The following statements are easily verified.

LEMMA 66.

$$(\nu+1)\,\sigma_{\nu+1}(x) = \sum_{p \leqslant x} \sigma_\nu(x/p) + \sum_{p^2 \leqslant x} \sigma_{\nu-1}(x/p^2) + \cdots + \sum_{p^{\nu+1} \leqslant x} \sigma_0(x/p^{\nu+1}).$$

(Pillai)

If corresponding to $F_2(s)$ of Lemma 4 we write $F'_\nu(s) = \sum_{Q_\nu} \frac{1}{Q_\nu^s}$,

where Q_ν is a number with ν prime factors, each factor being counted according to its multiplicity, then

$$\nu F'_\nu(s) = \left(\sum_p 1/p^s\right) F'_{\nu-1}(s) + \\ + \left(\sum_p 1/p^{2s}\right) F'_{\nu-2}(s) + \ldots + \sum_p 1/p^{\nu s}.$$

Similarly, equation 99 is true if all the terms are made positive and q_ν is replaced by Q_ν.

LEMMA 67. *Let* $A'(t) = \rho(t) + (-1)^{t-1}\lambda(t)$, $B'_0 = 1, B'_1 = B$ *and for* $r \geqslant 2$,

$$r B'_r = B'_{r-1} B + \sum_{t=2}^{r} (-1)^{t-1} A'(t) B'_{r-t},$$

$$\psi'(\nu, x) = \frac{x}{\log x} \sum_{r=0}^{\nu-1} B'_r \frac{(l_2 x)^{\nu-r-1}}{(\nu-r-1)!}.$$

Then

(i) $$f(z)\left\{\prod_p \left(1 - \frac{z^2}{p_2}\right)^{-1}\right\} = \exp\left\{B'_1 z - A'(2)\frac{z^2}{2} + A'(3)\frac{z^3}{3} - \ldots\right\}$$

$$= \sum_{r=0}^{\infty} B'_r z^r \quad \textit{for } |z| \leqslant 1/2,$$

where $f(z)$ *denotes the integral function of Lemma* 1. *Hence for* $|z| < 2$,

$$f(z)\left\{\prod_p \left(1 - \frac{z^2}{p^2}\right)^{-1}\right\} = \sum_{r=0}^{\infty} B'_r z^r.$$

(ii) *For* $1 \leqslant \nu \leqslant k\, l_2 x$, $k < 2$

$$\psi'(\nu, x) = \left\{1 + \theta\left(\frac{e_1}{l_2 x}\right)\right\} f(\omega)\left\{\prod_p \left(1 - \frac{\omega^2}{p^2}\right)^{-1}\right\} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x},$$

where e_1 *is a constant dependent upon* k *alone.*

(iii) *If* $1 \leqslant \nu \leqslant l_2 x$ *then*

$$\sigma_\nu(x) = \psi'(\nu, x) + \theta\left\{ e_2 \frac{x}{(\log x)^2} \frac{(l_2 x)^\nu}{\nu!} h_1(\nu)\right\}.$$

Hence, extending this result, we get

(iv) *If* $1 \leqslant \nu \leqslant k\, l_2 x, k < 2$ *then*

$$\sigma_\nu(x) = \left\{1 + \theta\left(\frac{e_3}{l_2 x}\right)\right\} f(\omega) \left\{\prod_p \left(1 - \frac{\omega^2}{p^2}\right)^{-1}\right\} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}$$

$e_3, e_4, \ldots$ *being positive constants dependent only upon* k. *From this* (6) *follows as a special case.*

The generating function of the coefficients B'_r has a pole at $z = 2$ while the corresponding function $f(z)$ in the case of $\pi_\nu(x)$ is an integral function.

LEMMA 68.

$$\begin{aligned}
&(\nu + 1)\,\rho_{\nu+1}(x) \\
&= \sum_{p \leqslant x} \rho_\nu(x/p) + \sum_{p^2 \leqslant x} \rho_\nu(x/p^2) + \ldots - \\
&\quad - \sum_{p^2 \leqslant x} \rho_{\nu-1}(x/p^2) - 2\sum_{p^3 \leqslant x} \rho_{\nu-1}(x/p^3) - 3\sum_{p^4 \leqslant x} \rho_{\nu-1}(x/p^4) - \ldots + \\
&\quad + \ldots + \\
&\quad + (-1)^{t-1} \sum_{\substack{p^r \leqslant x \\ r \geqslant t}} a_{t,r}\, \rho_{\nu+1-t}(x/p^r) + \\
&\quad + \ldots + \\
&\quad + (-1)^\nu \sum_{\substack{p^r \leqslant x \\ r \geqslant \nu+1}} a_{\nu+1,r},
\end{aligned}$$

where the coefficients $a_{t,r}$ *are given by the identity*

$$\frac{x^t}{(1-x)^t} = \sum_{r=t}^{\infty} a_{t,r}\, x^r.$$

LEMMA 69. *Let* $A''(t) = \zeta(t) + \sum_p \frac{1}{(p-1)^t}$, *for* $t \geqslant 2$,

$$B_0'' = 1,\ B_1'' = B + \sum_p \frac{1}{p\,(p-1)}\,.$$

For $r \geqslant 2$ let B_r'' be defined by

$$r B_r'' = B_{r-1}'' B_1'' + \sum_{t=2}^{r} (-1)^{t-1} A''(t) B_{r-t}'',$$

$$\psi''(\nu, x) = \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r'' \frac{(l_2 x)^{\nu-r-1}}{(\nu-r-1)!}.$$

Then

(i) $\sum_{r=0}^{\infty} B_r'' z_r$

$$= \exp\left\{ B_1'' z - A''(2)\frac{z^2}{2} + \ldots + (-1)^{t-1} A''(t) \frac{z^t}{t} + \ldots \right\}$$

$$= f(z) \exp\left[\sum 1/p(p-1) - \frac{z^2}{2}\left\{ \sum 1/(p-1)^2 - \sum 1/p^2 \right\} + \right.$$

$$\left. + \ldots + \frac{(-1)^{t-1}}{t}\left\{ \sum \frac{1}{(p-1)^t} - \sum \frac{1}{p^t} \right\} z^t \right]$$

$$= f(z). \exp\left[\log\left\{ \left(1 + \frac{z}{p-1}\right)\left(1 + \frac{z}{p}\right)^{-1} \right\} \right]$$

$$= f(z) \prod_p \left\{ 1 + \frac{z}{(p+z)(p-1)} \right\}, \text{ if } |z| \leqslant 1/2.$$

Hence for $|z| \leqslant 2$,

$$f(z) \prod_p \left\{ 1 + \frac{z}{(z+p)(p-1)} \right\} = \sum_{r=0}^{\infty} B_r'' z^r.$$

(ii) *For $1 \leqslant \nu \leqslant k\, l_2 x$, $k < 2$,*

$$\psi''(\nu, x) = \left\{ 1 + \theta\left(\frac{e_4}{l_2 x}\right) \right\} \times$$

$$\times f(\omega) \prod_p \left\{ 1 + \frac{\omega}{(p+\omega)(p-1)} \right\} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}.$$

(iii) *If $1 \leqslant \nu \leqslant l_2 x$ then*

$$\rho_\nu(x) = \psi''(\nu, x) + \theta\left\{ e_5 \frac{x}{(\log x)^2} \frac{(l_2 x)^\nu}{\nu!} h_2(\nu) \right\}.$$

Hence extending this result we get

(iv) *If* $1 \leqslant \nu \leqslant k\, l_2\, x$, $k < 2$, *then*

$$\rho_\nu(x) = \left\{1 + \theta\left(\frac{e_6}{l_2\, x}\right)\right\} f(\omega) \prod_p \left\{1 + \frac{\omega}{(p+\omega)(p-1)}\right\} \frac{(l_2\, x)^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}.$$

From this (7) *follows as a special case.*

In this case also, the generating function of the coefficients B_r'' has a pole at $z = 2$. Hence, it is highly probable that the results cannot be generalized for larger values of k by this method. Indeed, in the case of $\sigma_\nu(x)$ we shall presently see in the next chapter that an equation of this type does not hold when $\nu = [2 \log \log x] + 1$.

CHAPTER 10

MISCELLANEOUS RESULTS

In this chapter we shall give a very easy and direct proof of equation (9) and consider some deductions from our results. For the proof of (9) we need only Pillai's result (4).

LEMMA 70. *If* $\nu = [2 \log \log x] + 1$ *then when* $x \to \infty$,

$$\sigma_\nu(x)\ \Omega\ \frac{x}{\log x} \frac{(l_2\, x)^{\nu-1}}{(\nu-1)!}. \tag{345}$$

PROOF. For $m^2 \leqslant 2^{\nu^{1/3}+1}$, $F(m^2) \leqslant \nu^{1/3} + 1$.

Hence, by Lemma 51 when $l_2\, x \geqslant 10$,

$$\sigma_\nu(x) \geqslant \sum_{m^2 \leqslant 2^{\nu^{1/3}+1}} \pi_{\nu - F(m^2)}(x/m^2)$$

$$\geqslant H \sum_{m^2 \leqslant 2^{\nu^{1/3}+1}} \frac{x}{m^2 \log \overline{x/m^2}} \frac{(l_2\, \overline{x/m^2})^{\nu - F(m^2) - 1}}{(\nu - F(m^2) - 1)!} \quad \text{by (4).} \tag{346}$$

Since for $m^2 \leqslant 2^{\nu^{1/3}+1}$, $l_2 x \geqslant 10$,

$$l_2 \overline{x/m^2} = l_2 x + \log\left(1 - \frac{\log m^2}{\log x}\right)$$

$$\geqslant l_2 x + \log\left(1 - \frac{(\nu^{1/3}+1)\log 2}{\log x}\right)$$

$$> l_2 x + \log\left(1 - \frac{4\, l_2 x}{\log x}\right)$$

$$> l_2 x - 8\frac{l_2 x}{\log x} > l_2 x - \frac{1}{10}.$$

$$\frac{5}{2} l_2 \overline{x/m^2} > \frac{5}{2} l_2 x - \frac{1}{4} > 2\, l_2 x + 1 \geqslant \nu \geqslant \nu - F(m^2),\ e > 5/2.$$

From (346) it follows that when $x \geqslant x_0 > e^{e^{10}}$

$$\sigma_\nu(x) > \frac{H}{2}\frac{x}{\log x}\sum_{m^2 \leqslant 2^{\nu^{1/3}+1}} \frac{(l_2 x)^{\nu - F(m^2)-1}}{(\nu - F(m^2) - 1)!}\frac{1}{m^2}$$

$$> \frac{H}{4}\frac{x}{\log x}\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}\sum_{m^2 \leqslant 2^{\nu^{1/3}+1}} \frac{1}{m^2}\left(\frac{\nu}{l_2 x}\right)^{F(m^2)}$$

when $x \geqslant x_1 > x_0$ by virtue of (263),

$$\geqslant \frac{H}{4}\frac{x}{\log x}\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}\sum_{m^2 \leqslant 2^{\nu^{1/3}+1}} \frac{m^2}{2^{F(m^2)}}. \tag{347}$$

From (347) the result follows immediately since the series

$$\sum_m \frac{2^{(Fm^2)}}{m^2} > \sum_{r=1}^{\infty} \frac{2^{F(2^{2n})}}{2^{2n}}$$

is divergent.

LEMMA 71. *All the results about normal order of prime factors of a number n proved by Hardy and Ramanujan follow as particular cases of equations* (342)-(344).

Lemma 72. *Similarly we deduce the Pillai-Erdos results about numbers whose number of prime factors belong to a given arithmetical progression.*

Lemma 73. *Further improvements on the result of Pillai about maximum set $M(x)$ consisting of numbers $\leqslant x$ such that no number in the set is a multiple of any other in it by replacing the constant by unity.*

k	0	1	1.1	1 2	1 3	1 4	1.5	2
$f(k)$	1	$6/\pi^2$	.51	.47	42	.37	32	12
$f(k)\left\{\prod_p\left(1-\frac{k^2}{p^2}\right)^{-1}\right\}$	1	1	.92	.98	1.02	1.06	1.09	∞
$f(k)\left\{\prod_p\left(1+\frac{k}{(k+p)(p+1)}\right)\right\}$	1	1	.85	80	73	.66	.58	0 27

$f(k)$ increases steadily from 1 to the maximum value 1.03 as k increases from 0 to 0.14. It then decreases from 1 03 to $6/\pi^2$ as k increases from 0.14 to 1.

$f(k)\left\{\prod_p\left(1-\frac{k^2}{p^2}\right)^{-1}\right\}$ increases steadily from 1 to the maximum value 1.04 as k increases from 0.28 to 1.

$f(k)\left\{\prod_p\left(1+\frac{k}{(k+p)(p-1)}\right)\right\}$ increases from 1 to the maximum value 1.4 as k increases from 0 to 0 53. It then decreases from 1.4 to 1 as k increases from 0.53 to 1.

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NOTE ON A PAPER BY L. G. SATHE

By ATLE SELBERG

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While the results of Sathe's paper [*J. Indian Math. Soc.* 17 (1953), 63-141; 18 (1954), 27-81] are very beautiful and highly interesting, the way the author has proceeded in order to prove these results is a rather complicated and involved one, and this by necessity since a proof by induction after ν starting from the case $\nu = 1$, presents overwhelming difficulties in keeping track of the estimates of the remainder terms in their dependence of the two parameters ν and x.

A simpler and more natural way to obtain these results will be sketched briefly below. This approach also leads to a greater range of validity for some of Sathe's results. The natural starting point is the general divisor problem concerned with estimating the expression $\sum_{n \leq x} d_z(n)$, where

$$\sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} = \{\zeta(s)\}^z, \; d_z(1) = 1, \tag{1}$$

for $s = \sigma + it$, $\sigma > 1$. $\zeta(s)$ here is Riemann's Zeta function. The solution of this problem is well known if z is a rational integer, but seems never to have been stated in the literature for arbitrary complex z, though it would seem likely to have been known to specialists in the field, since the proof is so straightforward along classical lines. From this result, our Theorem 1, the rest follows without any difficulty.

In the following A is an arbitrary positive constant, and B a positive constant depending on the particular function $f(s, z)$, to occur later. The constants involved in the O's depend only on A in the case of Theorem 1, and only on the function $f(s, z)$ for the later theorems.

This note was originally intended for the editorial use of the Transactions of the American Mathematical Society.

THEOREM 1. *We have*

$$D_z(x) = \sum_{n \leqslant x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\{(x \log x)^{z-2}\},$$

uniformly for $|z| \leqslant A$, $x \geqslant 2$.

The proof proceeds along the standard lines, starting from the expression

$$\int_0 D_z(t)\, dt = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} \{\zeta(s)\}^z\, ds.$$

Deforming the path of integration using well-known estimates for $\log \zeta(s)$ in the neighbourhood of $\sigma = 1$, one obtains easily

$$\int_0^x D_z(t)\, dt = \frac{1}{2\pi i} \int_L \frac{x^{s+1}}{s(s+1)} \{\zeta(s)\}^z\, ds + O\left(\frac{x^2}{(\log x)^{3A+4}}\right), \tag{2}$$

where the path of integration consists of the line segment $\{\frac{1}{2} - i/(\log x),\ 1 - i/(\log x)\}$ the halfcircle with center at 1 joining $1 - i/(\log x)$ and $1 + i/(\log x)$ to the right of the line $\sigma = 1$ and the line segment $\{1 + i/(\log x),\ \frac{1}{2} + i/(\log x)\}$. Using the fact that for $y \geqslant x \geqslant 2$,

$$D_z(y) - D_z(x) = O\{(y - x)(\log y)^{|z|}\} + O(y^{1-1/(1+|z|)}),$$

which follows easily from the fact that $|d_z(n)| \leqslant d_k(n)$, where $k = [|z|] + 1$, one gets from (2),

$$\begin{aligned} D_z(x) &= \frac{1}{2\pi i} \int_L \frac{x^s}{s} \{\zeta(s)\}^z\, ds + O\left(\frac{x}{(\log x)^{A+2}}\right) \\ &= \frac{1}{2\pi i} \int_L \frac{x^s}{(s-1)^z} \frac{1}{s} \{(s-1)\,\zeta(s)\}^z\, ds + O\left(\frac{x}{(\log x)^{A+2}}\right) \\ &= \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\{x(\log x)^{z-2}\}, \end{aligned}$$

where the last equation follows easily from the regularity of $(s-1)\,\zeta(s)$ around $s = 1$, and the fact that the value at $s = 1$ is 1.

THEOREM 2. *Let*

$$f(s, z) = \sum_{n=1}^{\infty} \frac{b_z(n)}{n^s}, \quad \textit{for } \sigma > 1,$$

and let $\sum_{n=1}^{\infty} |b_z(n)| n^{-1} (\log 2n)^{B+3}$ *be uniformly bounded for* $|z| \leqslant B$.. *Further put*

$$\{\zeta(s)\}^z f(s,z) = \sum_{n=1}^{\infty} \frac{a_z(n)}{n^s}, \sigma > 1.$$

Then, we have

$$A_z(x) = \sum_{n \leqslant x} a_z(n) = \frac{f(1,z)}{\Gamma(z)} x(\log x)^{z-1} + O\{x(\log x)^{z-2}\},$$

uniformly for $|z| \leqslant B$, $x \geqslant 2$.

This follows from Theorem 1 on using the identity

$$A_z(x) = \sum_{n \leqslant x} b_z(n) D_z\left(\frac{x}{n}\right).$$

THEOREM 3. *Under the assumptions of the preceding theorem, and if* $a_z(n) = \sum_{k=0}^{\infty} c_k(n) z^k$ *for* $|z| \leqslant B$, *we have for* $k \geqslant 1$,

$$C_k(x) = \sum_{n \leqslant x} c_k(n) = \frac{x}{(k-1)! \log x} \left[\frac{d^{k-1}}{dz^{k-1}} \frac{f(1,z)}{\Gamma(1+z)} (\log x)^z\right]_{z=0} + \\ + O\left(\frac{x(\log\log x)^k}{k!(\log x)^2}\right),$$

uniformly for $k \leqslant B \log\log x$, $x \geqslant 3$.

This follows from Theorem 2. On noting that

$$C_k(x) = \frac{1}{2\pi i} \int_C \frac{A_z(x)}{z^{k+1}} dz, \qquad (3)$$

where C is a circle around the origin with the radius $k/\log\log x$, one obtains the estimate.

THEOREM 4.* *Under the assumptions of the preceding theorem, and if the second derivative of* $\dfrac{f(1,z)}{\Gamma(1+z)}$ *is uniformly bounded for*

*We could here replace $(k-1)/\log\log x$ by $k/\log\log x$ which would correspond more closely to Sathe's results; however the remainder term will then become worse if $k = o(\log\log x)$ as the factor $k/(\log\log x)^2$ is replaced by $1/\log\log x$.

$|z| \leqslant B$, *we have, for* $k \geqslant 1$,

$$C_k(x) = \frac{x}{\log x} \cdot \frac{f\left(1, \frac{k-1}{\log \log x}\right)}{\Gamma\left(1 + \frac{k-1}{\log \log x}\right)} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} +$$

$$+ O\left(\frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \cdot \frac{k}{(\log \log x)^2}\right),$$

uniformly for $k \leqslant B \log \log x$, $x \geqslant 3$.

We may assume $k > 1$, since it is otherwise trivially contained in the previous theorem. We then have only to evaluate the integral

$$\frac{1}{2\pi i} \int_C \frac{f(1,z)}{\Gamma(1+z)} \frac{(\log x)^z}{z^k} dz, \tag{4}$$

extended over the circle $|z| = \frac{k-1}{\log \log x}$. Writing $g(z)$ for $\frac{f(1,z)}{\Gamma(1+z)}$ and $r = \frac{k-1}{\log \log x}$, we have

$$\frac{1}{2\pi i} \int_C g(z) \frac{(\log x)^z}{z^k} dz$$

$$= \frac{g(r)}{2\pi i} \int_C \frac{(\log x)^z}{z^k} dz + \frac{1}{2\pi i} \int_C \{g(z) - g(r) - g'(r)(z-r)\} \frac{(\log x)^z}{z^k} dz$$

$$= g(r) \frac{(\log \log x)^{k-1}}{(k-1)!} + O\left(\int_C |z-r|^2 \left|\frac{(\log x)^z}{z^k}\right| |dz|\right).$$

A simple estimation of the last integral gives the desired result for the remainder term in Theorem 4.

The last two results contain all of Sathe's results by specialization as seen by taking $f(s, z)$ to be respectively

$$\prod_p \left(1 + \frac{z}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^z, \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z$$

and

$$\prod_p \left(1 - \frac{z}{p^s}\right)^{-1}\left(1 - \frac{1}{p^s}\right)^z.$$

$C_k(x)$ then respectively becomes $\pi_k(x)$, $\rho_k(x)$ and $\sigma_k(x)$ in Sathe's notation. It is easily seen that in the first two cases B may be chosen arbitrarily large whereas in the third case B has to be taken less than 2, but can be taken arbitrarily close to 2. This accounts for the breakdown of the asymptotic formula for $\sigma_k(x)$ in case $k = [2 \log \log x]$ for instance.

It should be noted that the range of Theorem 3 can be extended somewhat if we only want the remainder term to be of less order than the main term, since we can then integrate over the circle $|z| = B$ in (3) and obtain this result for $k \leqslant B^* \log \log x$, where B^* is a larger constant than B. Applying this to the case of $\sigma_k(x)$ where $f(1, z)$ has a pole of first order at $z=2$, but otherwise is regular for $|z| < 3$, we can obtain an asymptotic formula which is valid for $k \leqslant B^* \log \log x$, where B^* is an arbitrary constant > 2. This however has a more complicated form since it contains an additional term, which is due to the pole at $z = 2$. This term is of smaller order than the others as long as $k \leqslant (2 - \epsilon) \log \log x$, but becomes later dominating, which explains the change in character of the asymptotic behaviour of $\sigma_k(x)$, so that for $(2 + \epsilon) \log \log x \leqslant k \leqslant B^* \log \log x$ actually $\sigma_k(x) \sim C\,(x \log x)/2^k$, where C is the residue of $g(z)$ at $z = 2$

It should be noted that though Sathe's method is essentially elementary, the strong form of the prime-number theorem that he starts with cannot at present be obtained by elementary means.

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THEOREMS ON THE PRODUCT OF TWO SUMMABILITY METHODS WITH APPLICATIONS

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1. Introduction. In two successive papers [10,11] † O. Szász proved certain results on the product of two summability methods which with one exception, are included in Theorems I, II of this paper. Szász's results (with the exception referred to) are, in fact, the particular cases of Theorems I, II in which the $\mathscr{A}$-and the $\mathscr{B}$-transforms of these theorems are chosen as follows: (i) $\mathscr{A}\{s_n, x\}$ in Theorem I is the Abel transform of a sequence $s_n (n=0, 1, 2 \ldots)$, while $\mathscr{B}_n$ is the Cesaro transform of s_n of order $a > 0$ ([10], §2), or more generally, a regular Hausdorff transform [11]. (ii) $\mathscr{A}\{s_n, x\}$ in Theorem I is the Borel transform of s_n [11]. (iii) $\mathscr{A}\{s(x), t\}$ in Theorem II is the Laplace transform of a function $s(x)$, while $\mathscr{B}(x)$ is the Cesaro transform of $s(x)$ of order $a > 0$ ([10], §3).

Theorems I and II may be of some interest in themselves and also because they can be combined with standard Tauberian techniques, as shown in the concluding sections of this paper, to produce general Tauberian theorems, such as Theorem A, $\bar{\text{A}}$, which (in the case $k = 0$) give rise to a variety of other Tauberian theorems [6] analogous to certain theorems of Szász [9] for Abel-summable series. The theorems of Szász depend on the simplest theorem of the kind of Theorem A ([8], p. 52, Lemma), for Abel-summability, which he proved by using Theorem I in the form of Corollary I(a) with $k = 1$ in the same way as Theorem II is used to prove Theorem A.

The $\mathscr{A}$-transform of this paper is in general either a suitably restricted regular sequence-to-function transform or a suitably

† Szász's later paper [11] is not available to me. All my information about it is derived from *Math. Reviews* 15 (1954), p. 26.

restricted regular function-to-function transform, the resulting function being one of $x > 0$ in the first case and of $t > 0$ in the second case. The $\mathscr{B}$-transform is either a regular sequence-to-sequence Hausdorff transform or a regular continuous function-to-function Hausdorff transform, defined as usual in the following manner.

A regular Hausdorff transform of the sequence s_ν, $\nu = 0, 1, 2, \ldots$, is the sequence given by

$$\mathscr{B}\{s_\nu, n\} \equiv \mathscr{B}_n = \sum_{\nu=0}^{\infty} \lambda_{n,\nu} s_\nu, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where

$$\lambda_{n,\nu} = \begin{cases} \int_0^1 \binom{n}{\nu} u^\nu (1-u)^{n-\nu}\, d[\chi(u)] \text{ for } \nu \leqslant n, \\ 0 \text{ for } \nu > n, \end{cases}$$

$\chi(u)$ being a real function of bounded variation in $0 \leqslant u \leqslant 1$ such that

$$\chi(+0) = \chi(0) = 0, \ \chi(1) = 1.$$

$\mathscr{B}(s_\nu, n)$ can be either the Cesàro or the Hölder or the Euler transform, each of positive order, as well as certain less well-known logarithmic and exponential transforms ([4], §§ 11.12, 11.13, 11.14).

A regular continuous Hausdorff transform of the function $s \equiv s(x)$, supposed to be Borel-measurable, is the function given by the integral †

$$\mathscr{B}\{s, x\} \equiv \mathscr{B}(x) = \int_0^1 s(xu)\, d[\chi(u)], \tag{2}$$

where $\chi(u)$ is the function involved in the definition of the $\lambda_{n,\nu}$ of (1).

The most important special case of $\mathscr{B}(x)$ is the Cesàro transform of $s(x)$ of order $\alpha > 0$ ([4], § 11.18), corresponding to

$$\chi(u) = 1 - (1-u)^\alpha.$$

† All the integrals of this paper, unless otherwise stated, are either Lebesgue or Lebesgue-Stieltjes integrals.

2. A theorem on the product of a Hausdorff sequence-to-sequence transform and a power-series transform. A consequence of the usual condition for one regular Hausdorff method of summing sequences to include another such method ([4]. § 11.10), indicated by M. S. Ramanujan ([7], Theorem 5 and Corollary), is that *if $\mathscr{B}$, $\mathscr{B}'$ are two regular sequence-to-sequence methods of Hausdorff summability, then $\mathscr{B}'$-summability of a sequence s_n to sum S implies $\mathscr{B}'$ $\mathscr{B}$-summability of s_n to sum S, i.e. in our notation, $\mathscr{B}'\{s_\nu, n\} \equiv \mathscr{B}'_n \to S$ implies $\mathscr{B}'\{\mathscr{B}_\nu, n\} \to S$, provided that none of the moment constants of the methods $\mathscr{B}$, $\mathscr{B}'$ vanishes.* The following result is another property of regular sequence-to-sequence Hausdorff transforms, which, though deeper than the foregoing, can be placed in the same context.

Theorem I. *Let $\mathscr{A}$ and $\mathscr{B}$ denote two regular summability methods for sequences. Let $\mathscr{A}\mathscr{B}$ denote the iteration product which associates with a given sequence s_n the $\mathscr{A}$-transform of its $\mathscr{B}$-transform. Then $\mathscr{A}$-summability of s_n to sum S implies $\mathscr{A}\mathscr{B}$-summability to sum S in all cases in which $\mathscr{B}$,$\mathscr{A}$ are defined as follows.*

$\mathscr{B}$-transform of $s_n = \mathscr{B}_n$ defined by (1);

$\mathscr{A}$-transform of s_n is the regular power-series transform :

$$\mathscr{A}\{s_n, x\} = \frac{\sum_0^\infty p_n x^n s_n}{\sum_0^\infty p_n x^n}, \quad p_n \geqslant 0,\ 0 < x < \rho, \tag{3}$$

where $\rho \leqslant \infty$ is the radius of convergence of the power series $f(x) \equiv \Sigma p_n x^n$, $\Sigma p_n \rho^n = \infty$, and $\mathscr{A}$-summability of s_n to sum S is defined by

$$\mathscr{A}\{s_n, x\} \to S \text{ as } x \to \rho - 0,$$

and is supposed to carry with it the implication that, for every u such that $0 < u < 1$,

C′ (i) $\mathscr{A}_u\{s_n, x\} \equiv \dfrac{\sum_0^\infty \dfrac{f^{(n)}(x - xu)}{n!} x^n u^n s_n}{f(x)}, \ 0 < x < \rho,$

is (absolutely) convergent,

(ii) $\mathscr{A}_u\{s_n, x\} \ll F(x), \ 0 < x < \rho,$

(iii) $\mathscr{A}_u\{s_n, x\} \to S$ *as* $x \to \rho - 0,$

(iv) $|\mathscr{A}_u\{s_n, x\}| \leqslant M$ *(a constant).*

[In regard to C′ (i) it may be pointed out that, for every u in question, convergence of $\mathscr{A}_u\{s_n, x\}$ for any x, $0 < x < \rho$, implies absolute convergence exactly as convergence of a power series implies absolute convergence.]

Proof. Let $0 < x < \rho$. With this restriction, replacing s_n in $\mathscr{A}\{s_n, x\}$ as defined in (3) by $\mathscr{B}_n$, $n = 0, 1, 2, \ldots$, as defined in (1), we obtain

$$f(x)\mathscr{A}\{\mathscr{B}_n, x\} = \sum_0^\infty p_n x^n \mathscr{B}_n$$

$$= \sum_0^\infty p_n x^n \sum_{\nu=0}^n s_\nu \int_0^1 \binom{n}{\nu} u^\nu (1 - u)^{n-\nu} \, d[\chi(u)] \tag{4}$$

$$= \int_0^1 \sum_0^\infty p_n x^n \sum_{\nu=0}^n s_\nu \binom{n}{\nu} u^\nu (1-u)^{n-\nu} \, d[\chi(u)]. \tag{5}$$

To justify the equality of the functions in (5) and (4), and, in fact, to prove the existence of $\mathscr{A}\{\mathscr{B}_n, x\}$ through that of the function in (5), it is sufficient to assume (as we may) that $\chi(u)$ is monotonic increasing and to establish the similar equality

$$\sum_0^\infty p_n x^n \sum_{\nu=0}^n |s_\nu| \int_0^1 \binom{n}{\nu} u^\nu (1 - u)^{n-\nu} \, d[\chi(u)]$$

$$= \int_0^1 \sum_0^\infty p_n x^n \sum_{\nu=0}^n |s_\nu| \binom{n}{\nu} u^\nu (1 - u)^{n-\nu} \, d[\chi(u)], \tag{6}$$

where $|s_\nu|$ takes the place of s_ν. This equality (6) is valid because its right-hand integral is finite on account of the integrand

being expressible, after an interchange of summations, as the function

$$\sum_{0}^{\infty} \frac{f^{(n)}(x - xu)}{n!} x^n u^n |s_n| \equiv f(x)\mathscr{A}_u \{|s_n|, x\},$$

which is Borel-measureable and dominated by $f(x) F(x)$ on account of hypotheses C′ (i), (ii). Consequently the function in (5) exists and is expressible in the form (4), i.e.

$$f(x)\mathscr{A}\{\mathscr{B}_n, x\} = \int_0^1 \sum_{0}^{\infty} \frac{f^{(n)}(x - xu)}{n!} x^n u^n s_n \, d[\chi(u)],$$

or

$$\mathscr{A}\{\mathscr{B}_n, x\} = \int_0^1 \mathscr{A}_u \{s_n, x\} \, d[\chi(u)]. \tag{7}$$

By using hypotheses C′ (iii), (iv) in (7), and appealing to the Lebesgue theorem of bounded convergence for taking limits under integral signs, we obtain the desired conclusion .

$$\mathscr{A}\{\mathscr{B}_n, x\} \to S \text{ as } x \to \rho - 0.$$

COROLLARY I. *In the* $\mathscr{A}$*-transform* (3) *of Theorem* I, p_n *[and* $f(x)$*] can be chosen in one of the following ways*:

(*a*) $p_n = (-1)^n \binom{-k}{n}$, $k > 0$, [*i.e.* $f(x) = (1 - x)^{-k}, \rho = 1$],

(*b*) $p_n = 1/n!$, [*i.e.* $f(x) = e^x, \rho = \infty$].

The choice (*a*) of p_n with $k = 1$ makes $\mathscr{A}$-summability Abel summability while the choice (*b*) makes $\mathscr{A}$- summability Borel summability.

To prove Corollary I we argue as follows.

In both cases (*a*) and (*b*), it is easy to verify (by direct calculation) that, for $0 < u < 1$,

$$\mathscr{A}_u \{s_n, x\} = \mathscr{A}\{s_n, \xi\}, \quad \xi = \frac{\rho x u}{\rho - x + xu} \; (< x),$$

where ξ is to be interpreted as xu in case (*b*) where $\rho = \infty$.

In either case conditions C′ (i), (iii) clearly hold and condition C′ (ii) can be shown to hold in the form

$$\mathscr{A}_u\{\,|s_n|, x\} = \mathscr{A}\{\,|s_n|, \xi\} < \mathscr{A}\{\,|s_n|, \xi\}\, f(\xi) < \mathscr{A}\{\,|s_n|, x\}\, f(x).$$

To show that condition C′ (iv) holds in the case (*a*) in which $\rho = 1$, we choose δ, $0 < \delta < 1$, so that

$$|\mathscr{A}\{s_n, \xi\}| \leqslant M_1 \text{ for } 1 - \delta \leqslant \xi < 1, \tag{8}$$

this being possible in virtue of the fact that $\mathscr{A}\{s_n, \xi\} \to S$ as $\xi \to 1 - 0$. Then, writing

$$\xi = \frac{xu}{1 - x + xu}, \; u_1 = \left(\frac{1}{\delta} - 1\right)\left(\frac{1}{x} - 1\right), \; 0 < x < 1, u > 0,$$

we have

$$|\mathscr{A}_u\{s_n, x\}| \equiv |\mathscr{A}\{s_n, \xi\}| \leqslant \begin{cases} M_1 \text{ for } 1 - \delta \leqslant \xi < 1, \text{ or } u_1 \leqslant u, \\ M_2 \text{ for } 0 < \xi < 1 - \delta, \text{ or } u < u_1, \end{cases} \tag{9}$$

where the first inequality is merely (8) and the second is a consequence of the fact that $\mathscr{A}\{s_n, \xi\}$ is the quotient of two power series in ξ of which the series in the numerator has convergence-radius not less than 1 and the series in the denominator has convergence-radius 1 and least value non-zero (when $\xi = 0$). Thus in the case (*a*), the two inequalities of (9) together establish C′(iv) when $u_1 < 1$ while the first inequality alone is sufficient for this purpose when $u_1 \geqslant 1$. In the case (*b*), a similar reasoning leads to C′(iv); only we have to choose $X > 0$ so that we have

$$|\mathscr{A}\{s_n, \xi\}| \leqslant M_1 \text{ for } \xi \geqslant x,$$

instead of (8), and write $\xi = x\,u$, $u_1 = X/x$ ($x > 0$, $u > 0$), modifying (9) accordingly.

Remarks on Theorem I. In Theorem I, a power series $f(x) = \Sigma\, p_n\, x^n$ other than the series of Corollary I, which can be used for defining the $\mathscr{A}$-transform (3), is $f(x) = \log(e/1 - x)$, $|x| < 1$. But not all power series (with non-negative coefficients) are suitable for this purpose as we can see from the example

$$f(x) = (1 - x^2)^{-1},\; |x| < 1,\; s_n = \frac{1 + (-1)^n}{2},$$

where it is easily verified that

$$\mathscr{A}\{s_n, x\} = 1. \text{ if } |x| < 1, \quad \lim_{r \to 1-0} \mathscr{A}_u\{s_n, x\} = \tfrac{1}{2}, \text{ for } 0 < u < 1.$$

i.e. that condition C′ (iii) of Theorem I is not fulfilled.

3. A theorem on the product of a Hausdorff function-to-function transform and an integral transform.

THEOREM II. *Let $\mathscr{A}$, $\mathscr{B}$ denote two regular summability methods for a function $s(x)$ which is Borel-measurable in every finite interval of $x \geqslant 0$. Let $\mathscr{A}\,\mathscr{B}$ denote the iteration product which associates with a given function $s(x)$ the $\mathscr{A}$-transform of its $\mathscr{B}$-transform. Then $\mathscr{A}$-summability of $s(x)$ to sum S implies $\mathscr{A}\mathscr{B}$-summability to S in all cases in which $\mathscr{B}$, $\mathscr{A}$ are defined as follows.*

$\mathscr{B}$-transform of $s(x) = \mathscr{B}(x)$ defined by (2) ;

$\mathscr{A}$-transform of $s(x)$ is the regular integral transform

$$\mathscr{A}\{\, s(x), t\} = t \int_0^\infty \psi(xt)\, s(x)\, dx, \quad t > 0, \tag{10}$$

where $\psi(x)$ is a positive monotonic decreasing function of $x \geqslant 0$ and $\mathscr{A}$-summability to sum S is defined by

$$\mathscr{A}\{s(x), t\} \to S \text{ as } t \to +0, \tag{11}$$

with the additional condition that

C″ (i) *$\mathscr{A}\{s(x), t\}$ is (absolutely) convergent for every $t > 0$,*

(ii) *$|\mathscr{A}\,\{s(x), t\}| \leqslant M$. for $0 < t < \infty$*

PROOF. For $t > 0$, we obtain from the definition of $\mathscr{B}\{s(x)\}$ in (2),

$$\mathscr{A}\{\mathscr{B}(x), t\} = t \int_0^\infty \psi(xt)\, \mathscr{B}(x)\, dx$$

$$= t \int_0^\infty \psi(xt)\, dx \int_0^1 s(xu)\, d\,[\chi(u)] \tag{12}$$

$$= t \int_0^\infty \psi(xt)\, dx \int_0^x s(v)\, d\,[\chi(v/x)] \tag{13}$$

$$= t \int_0^\infty s(v)\, dv \int_v^\infty \psi(xt)\, d\,[\chi(v/x)]. \tag{14}$$

The equality of the repeated integrals in (14) and (13), which, in fact, establishes the existence of $\mathscr{A}\{\mathscr{B}(x), t\}$ from that of the repeated integral (14), is justified by Fubini's theorem†. For, supposing (as we may) that χ is monotonic increasing, we see that the existence of

$$t\int_0^\infty |s(v)|\,dv\int_v^\infty \psi(xt)\,|d[\chi(v/x)]|,$$

is ensured by the relations

$$\int_v^\infty \psi(xt)\,|d[\chi(v/x)]| < \psi(vt)\,[\chi(1)-\chi(+0)],$$

$$t\int_0^\infty |s(v)|\,dv\,\psi(vt) < \infty,$$

of which the former is a result of $\psi(x)$ being monotonic decreasing and the latter is hypothesis C″ (i). Hence we have, changing the order of integration in (12),

$$\begin{aligned}\mathscr{A}\{\mathscr{B}(x), t\} &= \int_0^\infty d[\chi(u)]\,t\int_0^\infty \psi(xt)\,s(xu)\,dx\\ &= \int_0^1 d[\chi(u)]\,\frac{t}{u}\int_0^\infty \psi\left(x'\frac{t}{u}\right)s(x')\,dx'\\ &= \int_0^1 d[\chi(u)]\,\mathscr{A}\{s(x'), t/u\}. \qquad (15)\end{aligned}$$

Now, from the hypothesis of $\mathscr{A}$-summability of $s(x')$ and the hypothesis C″ (ii), we have, for every u such that $0<u\leqslant 1$,

$$\mathscr{A}\{s(x'), t/u\} \to S \text{ as } t\to +0,$$

$$|\mathscr{A}\{s(x'), t/u\}| \leqslant M.$$

Therefore an application of Lebesgue's theorem of bounded convergence to the integral in (15) establishes the conclusion sought, viz.

$$\mathscr{A}\{\mathscr{B}(x), t\} \to S \text{ as } t\to +0.$$

†I am indebted to Dr. V Ganapathy Iyer for pointing out to me that a condition omitted in Widder's statement of Fubini's theorem ([12], p. 26) is required for the truth of the theorem (e.g. [3], p. 148) The condition in the particular case of (12) is that $t\,\psi(xt)\,s(xu)$ should be measurable with respect to the product measure of the Lebesgue measure [x-axis] and the Lebesque-Stieltjes measure determined by $\chi(u)$ [u-axis]. It is fulfilled because of the assumption of Borel-measurability of $s(u)$.

REMARK ON THEOREM II. From (15) and hypothesis C″ (ii) of Theorem II we get the supplementary conclusion that $\mathscr{A}\{\mathscr{B}\{s(x)\}, t\}$ is bounded for $0 < t < \infty$. Hence *conditions* C″ (i), (ii) *and* (11) *of Theorem* II *hold when* $s(x)$ *in* $\mathscr{A}\{s(x), t\}$ *is replaced by the result of any finite succession of continuous Hausdroff transformations of* $s(x)$.

4. Applications to integral transforms of functions. Theorem II combined with the following 'positive' Tauberian theorem yields two useful results which appear as Theorems A, B in the sequel.

LEMMA 1. *Suppose that* $p(x) \geqslant 0$ *for* $x \geqslant 0$ *and*

$$\mathscr{A}\{p(x), t\} = t\int_0^\infty \psi(xt)\, p(x)\, dx, \quad t > 0,$$

where $\psi(x)$, *in addition to being positive monotonic decreasing, is such that*

$$\int_0^\infty \psi(x)\, dx = 1, \int_0^\infty x^{iu}\, \psi(x)\, dx \neq 0\, (-\infty < u < \infty). \tag{16}$$

Then

$$\lim_{x\to\infty} \frac{1}{x}\int_0^\infty p(u)\, du = \lim_{t\to+0} \mathscr{A}\{p(x), t\},$$

whenever the limit on the right side exists.

Lemma 1 is a particular case of a theorem of Hardy ([4], Theorem 236). There is a version of the lemma which can be used to prove the prime number theorem ([5], Remarks (ii), (iii) on Theorem 1).

THEOREM A. *Suppose that* $\mathscr{A}\{s(x), t\}$ *is defined as in* (10) *and satisfies conditions* C″ (i), (ii) *and* (11) *of Theorem* II. *Suppose that the kernel* $\psi(x)$, *besides being positive monotonic decreasing and an integral, satisfies* (16) *along with the further condition*

$$\psi(x) \log x \in L(0, \infty), \tag{17}$$

which, if we set

$$\phi(x) = \int_x^\infty \psi(u)\, du,$$

can be expressed in the form

$$\int_0^1 \frac{1-\phi(x)}{x}\,dx < \infty, \int_1^\infty \frac{\phi(x)}{x}\,dx < \infty. \tag{17'}$$

Then the hypothesis

$$\delta_k(x) \equiv (k+1)\,\{\,C_k(x) - C_{k+1}(x)\,\} = O_L(1), k \geqslant 0,\ x \to \infty, \tag{18}$$

where

$$C_a(x) = a \int_0^x s(u)\left(1-\frac{u}{x}\right)^{a-1}\frac{du}{x},\ a > 0,\ C_0(x) = s(x),$$

implies the conclusion

$$C_{k+1}(x) \to S \ \text{as}\ x \to \infty. \tag{19}$$

Proof. Choosing, in (2), $\chi(u) = 1-(1-u)^a$, $a > 0$, we have †

$$\mathscr{B}(x) = \int_0^1 s(xu)\,d\,[1-(1-u)^a] \equiv C_a(x).$$

The above choice of $\mathscr{B}(x)$ in Theorem II gives us

$$\mathscr{A}\{C_a(x), t\} \equiv t\int_0^\infty \psi(xt)\,C_a(x)dx \to S \ \text{ as }\ t \to +0,\ a > 0. \tag{20}$$

this relation being true for $a = 0$ also by hypothesis. In (20), putting $a = k$ and $a = k+1$ successively, we obtain

$$\mathscr{A}\{C_{k+1}(x), t\} - \mathscr{A}\{C_k(x), t\} \equiv t\int_0^\infty \psi(xt)\frac{\delta_k(x)}{k+1}\,dx \to 0 \ \text{ as }\ t \to +0,$$

or

$$t\int_0^\infty \psi(xt)\,\{\,\delta_k(x) + K\}\,dx \to K \ \text{ as }\ t \to +0,$$

where K is a constant chosen so that $\delta_k(x) + K \geqslant 0$, this choice being possible by (18). Hence, taking $p(x) = \delta_k(x) + K$ in Lemma 1, we get

$$\frac{1}{x}\int_0^x \{\delta_k(u) + K\}\,dx \to K,\ \text{ as } x \to \infty,$$

† It is clear that, with the choice of $\chi(u)$ we have made, the condition of Borel-measurability of $s(x)$ assumed in (2) may be replaced by that of Lebesgue-measurability as usual.

which can be written

$$\delta^*_k(x) \equiv \frac{1}{x}\int_0^x \delta_k(u)\,du = o(1), x \to \infty. \tag{21}$$

Now, from the definitions of $\delta^*_k(x)$, $\delta_k(x)$ we have

$$x\,\delta^*_k(x) = \int_0^x u\,C'_{k+1}(u)\,du = x\,C_{k+1}(x) - \int_0^x C_{k+1}(u)\,du,$$

$$\int_0^x \frac{\delta^*_k(u)}{u}\,du = \frac{1}{x}\int_0^x C_{k+1}(u)\,du \equiv C^*_{k+1}(x). \tag{22}$$

Consequently

$$\begin{aligned}\mathscr{A}\{C^*_{k+1}(x), t\} &\equiv t\int_0^\infty \psi(xt)\,C^*_{k+1}(x)\,dx \\ &= t\int_0^\infty \psi(xt)\,dx \int_0^\infty \frac{\delta^*_k(u)}{u}\,du \\ &= t\int_0^\infty \frac{\delta^*_k(u)}{u}\,du \int_u^\infty \psi(xt)\,dx = \int_0^\infty \frac{\delta^*_k(u)}{u}\,\phi(ut)\,du.\end{aligned} \tag{23}$$

The repeated integrals in the last two lines are equal by Fubini's theorem whose use is justified by the fact

$$t\int_0^\infty \frac{|\delta^*_k(u)|}{u}\,du \int_u^\infty \psi(xt)\,dx = \int_0^\infty \frac{|\delta^*_k(u)|}{u}\,\phi(ut)\,du < \infty,$$

following from (21) and the second condition of (17′). From (23) and (22) we get, by subtraction,

$$\begin{aligned}\mathscr{A}\{C^*_{k+1}(x), t\} - C^*_{k+1}(x) &= \int_0^x \frac{\phi(ut)-1}{u}\,\delta^*_k(u)\,du + \\ &\qquad + \int_x^\infty \frac{\phi(ut)}{u}\,\delta^*_k(u)\,du \\ &= \int_0^{xt} \frac{\phi(u)-1}{u}\,\delta^*_k(u/t)\,du + \\ &\qquad + \int_{xt}^\infty \frac{\phi(u)}{u}\,\delta^*_k(u/t)\,du.\end{aligned} \tag{24}$$

In (24) choose $t = 1/x$ and let $x \to \infty$. Then both the integrals on the right side of (24) tend to 0, as we can see from (17′) and an appeal to

Fatou's theorem in which we use (21). Furthermore $\mathscr{A}\{C^*_{k+1}(x), t\}$ on the left side of (24) tends to S since it is obtained from $\mathscr{A}\{s(x), t\}$ by a change of $s(x)$ to $C^*_{k+1}(x)$ involving two successive Hausdorff transformations of $s(x)$.

Therefore (24) leads to the relation

$$C^*_{k+1}(x) \equiv \frac{1}{x}\int_0^x C_{k+1}(u)\, du \to S \text{ as } x \to \infty.$$

This relation in turn leads to the desired conclusion (19) when we use it in conjunction with (18) in the form $x\, C'_{k+1}(x) = O_L\,(1)$, appealing to a classical Tauberian theorem for Cesaro summability.

The following result, implicit in the proof of Theorem A, is well-known in the case $k = 0$, $\psi(u) = e^{-u}$ ([2], Theorem 3.81.)

COROLLARY A. *In the notation of Theorem* A, *necessary and sufficient conditions for* $C_k(x)$, $k \geqslant 0$, *to tend to* S *as* $x \to \infty$ *are that* (i) $\mathscr{A}\{C_k(x), t\}$ *should exist for* $0 < t < \infty$ *tending to* S *as* $t \to +0$, (ii) $\delta_k(x)$ *should tend to* 0 *as* $x \to \infty$.

In connection with Corollary A it may be pointed out that the condition (i) of the corollary and the condition $\delta_k(x) = O(1)$ as $x \to \infty$ together imply (by a well-known Tauberian theorem) that $C_k(x) = O(1)$ and hence that $\mathscr{A}\{C_k(x), t\}$ is bounded for $0 < t < \infty$.

For,

$$C_k(x') - C_k(x) = \frac{[\delta_k(x') - \delta_k(x)]}{(k+1)} + \int_x^{x'} \frac{\delta_k(u)}{u}\, du,$$

i e.

$$C_k(x') - C_k(x) = O(1)\,[1 + \log \lambda],\ x \leqslant x' \leqslant \lambda x,\ x \to \infty.$$

The next theorem is a combination of Theorem II and Lemma 1, similar to Theorem A but simpler. It has an analogue for the Abel and the Cesaro transforms of a sequence, due to Hardy and Littlewood, which is a similar combination of Theorem I in the form of Corollary I(a) with $k = 1$ and a 'positive' Tauberian theorem for Abel-summable sequences.

THEOREM B. *Theorem* A *can be restated with* (17) *omitted,* (18) *replaced by* $C_k(x) \geqslant 0$, *and no other change.*

PROOF. When $k = 0$ Theorem B reduces to Lemma 1. When $k > 0$ we put $a = k$ in (20) which is still true and appeal to Lemma 1 with $p(x) = C_k(x)$, obtaining

$$\frac{1}{x}\int_0^x C_k(u)\,du \to S \text{ as } x \to \infty.$$

This relation can be written, after an integration by parts, in the form

$$\frac{1}{k+1}\,C_{k+1}(x) + \frac{k}{k+1}\,\frac{1}{x}\int_0^x C_{k+1}(u)\,du \to S,$$

whence it follows that $C_{k+1}(x) \to S$ by an integral analogue of Mercer's theorem.

NOTE. (i) In Theorems II, A, B we can have one of the following three special choices for $\psi(u)$:

$$\psi(u) = e^{-u};\ \psi(u) = \frac{\rho}{(1+u)^{\rho+1}},\ \rho > 0;\ \psi(u) = -\frac{d}{du}\left(\frac{u}{e^u - 1}\right),$$

which in turn and in the order of special $\psi(u)$ mentioned, make $\mathscr{A}\{s(x), t\}$ the Laplace transform of $s(x)$, the Stieltjes transform and the Lambert transform.

(ii) In the case of the Laplace transform, Theorem B may be deduced from Theorem A which itself is deducible from a more general theorem ([1], Theorem II′) containing also a theorem of Hardy and Littlewood. The proof of Theorem A given here follows the proof, given by Chandrasekharan and Minakshisundaram ([2], pp. 94-5), for Szász's special case of the theorem in which $\mathscr{A}\{s(x), t\}$ is the Laplace transform and $k = 0$.

5. Applications to integral transforms of a series. As a preliminary to such applications we state the following known result

([5], Lemma 2) which extends a theorem on the Laplace transform ([12], Chap. II, Theorem 2.3*a*).

LEMMA 2. *Let $s(x)$ be of bounded variation in every finite interval of $x \geqslant 0$ and let $s(x) = 0$. If, for $t > 0$, a transform of $s(x)$ is defined by*

$$\bar{\mathscr{A}}\{s(x),\, t\} = \int_0^\infty \phi(xt)\, d\,[s(x)],\ \textit{where}\ \phi(x) = \int_x^\infty \psi(u)\, du, \tag{25}$$

and $\psi(x)$ is positive monotonic decreasing, then $\bar{\mathscr{A}}\{s(x), t\} = \mathscr{A}\{s(x), t\}$ as defined in (10).

With the help of Lemma 2, Theorems A, B can be restated as follows for the transform of $s(x)$ in (25).

THEOREMS $\bar{\text{A}}$, $\bar{\text{B}}$. *In Theorems* A, B *we can replace $\mathscr{A}\{s(x), t\}$ by $\bar{\mathscr{A}}\{s(x), t\}$, supposing that the latter integral converges for $t > 0$ and tends to S as $t \to +0$.*

PROOF. $\mathscr{A}\{s(x), t\} = \bar{\mathscr{A}}\{s(x), t\} \to S$ as $t \to +0$, and so there is a $t_1 > 0$ such that $\mathscr{A}\{s(x), t\}$ is bounded for $0 < t < t_1$; also, for $t \geqslant t_1$,

$$\begin{aligned} |\mathscr{A}\{s(x),\, t\}| &= |\bar{\mathscr{A}}\{s(x), t\}| \\ &\leqslant \int_0^\infty \phi(xt)\,|d\,[s(x)]| \\ &\leqslant \int_0^\infty \phi(xt_1)\,|d\,[s(x)]|. \end{aligned}$$

Hence $\mathscr{A}\{s(x), t\}$ satisfies (11) and C″ (i), (ii) posited in Theorems II, A, B; and Theorems $\bar{\text{A}}$, $\bar{\text{B}}$ follow from Theorem A, B.

In Theorems $\bar{\text{A}}$, $\bar{\text{B}}$, $\bar{\mathscr{A}}\{s(x), t\}$ becomes the series:

$$\left.\begin{aligned} \bar{\mathscr{A}}\{s(x), t\} &= \sum_{n=1}^{\infty} a_n\, \phi(\lambda_n t),\ t > 0, \\ 0 < \lambda_1 < \lambda_2 < \ldots,&\ \ \lambda_n \to \infty, \end{aligned}\right\} \tag{25′}$$

when we define $s(x)$ as the λ_n-step function

$$s(x) = \begin{cases} a_1 + a_2 + \ldots + a_n \text{ for } \lambda_n \leqslant x < \lambda_{n+1}, n \geqslant 1. \\ 0 \text{ for } 0 \leqslant x < \lambda_1. \end{cases} \tag{25''}$$

A deduction from the case $k = 0$ of Theorem $\bar{\mathrm{A}}$ ([6], Corollary III′) deserving to be recalled may be stated here.

COROLLARY $\bar{\mathrm{A}}$.† *Let* $\bar{\mathscr{A}}\{s(x), t\}$ *be defined by* (25) *and* $s(x)$ *by* (25″).

Assume the two conditions: $\bar{\mathscr{A}}\{s(x), t\} \to S$ *as* $t \to +0$ *and*

$$\lim_{\delta\to+0} \varliminf_{n\to\infty} \underline{\operatorname{bound}}_{\lambda_n < x < (1+\delta)\lambda_n} [s(x) - s(\lambda_n)] \geqslant 0,$$

with EITHER $\lim \lambda_{n+1}/\lambda_n = 1$ OR $\varliminf a_n \geqslant 0$ *Then*

$$\sum_{n=1}^{\infty} a_n = S.$$

The special case of Corollary $\bar{\mathrm{A}}$ in which ψ is the Lambert kernel and $\lambda_n = n$ is important; and an interesting indirect proof has been suggested for it by Hardy ([4] Appendix IV, Theorem 261.)

NOTE. In propositions such as the one last stated, when $\psi(u)$ is either the Laplace or the Lambert kernel, the transform $\bar{\mathscr{A}}\{s(x), t\}$ of the $s(x)$ of (25″) can be assumed to be a Riemann-Stieltjes integral which is (non-absolutely) convergent for $t > 0$. For, in the first place, this assumption implies that $s(x)\,\phi(x\epsilon) \to 0$ as $x \to \infty$ for every $\epsilon > 0$ ([5], Lemma 1) and thence in the case of the Lambert transform as in the case of the Laplace transform ([12], Chap. II, Theorem, 2.3a), that $\mathscr{A}\{s(x), t\}$ is absolutely convergent for $t > 0$. Secondly, $\bar{\mathscr{A}}\{s(x), t\}$ is (i) bounded near $t = 0$ being convergent as $t \to +0$, (ii) bounded for finite t being a continuous function of t by uniform convergence, (iii) bounded as $t \to \infty$ by an argument for the Laplace transform ([12], Chap. V, Theorem 1, case $\gamma = 0$) applicable also to the Lambert transform.

† It may be pointed out here that, in the paper [6] where Corollary $\bar{\mathrm{A}}$ is proved, there is a related result ([6], p. 383, Theorem III) whose enunciation contains the misprint $\lambda_{n+1}/\lambda_n \to \infty$ which should be corrected to $\lambda_{n+1}/\lambda_n \to 1$.

(*Added in proof*). After this paper was submitted for publication I received, by courtesy of its author, a reprint of an article [1a] which proves directly Corollary I(a) for any $k > 0$ and bases its main result ([1a]. Theorem (8.4)) on the corollary.

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A NOTE ON TYPICAL MEANS

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1. This note has for its object the clarification of certain points in our book [1]. This clarification seems to us to be necessary in order to remove possible misconceptions regarding the validity of some of the results in [1] which are discussed in [2]; it will appear that those results are correct. We take this opportunity also to touch on some points other than those arising from [2]. Chapter and page references, unless otherwise indicated, apply to the book.

2. It has been suggested [2] that in Chapter I, §1.8 has certain "shortcomings", since no indication has been given in the enunciation of Theorem 1.81 that the function $A(x)$ occurring in the hypothesis satisfies a wider definition than the one we started with on page 1, such an indication being necessary to make the result "significant". The correctness of the result, with the given definition, has, however, not been questioned.

We wish to point out, in the first place, that we did indicate, in the *notes* (p. 31) on an earlier section (§1.2), the possibility of widening the definition without affecting the results. We said that $A(x)$ could be any function of bounded variation in every finite interval instead of being a step-function of the kind defined at the very beginning. This remark was intended to be carried over *wherever necessary and wherever possible* in Chapter I, and in particular to §1.8 and Theorem 1.81. This settles the point raised.

In the second place, we may remark that even if the definition of $A(x)$ is not widened as indicated, significant conclusions do emerge from Theorem 1.81, *in certain cases*. Instead of the definition

$$A(x) = \sum_{\lambda_n \leqslant x} a_n \tag{1}$$

that we begin with, we can consider an 'equivalent' definition:

$$A'(x) = \sum_{\lambda_n < x} a_n. \tag{2}$$

The first gives a step-function which is right-continuous at $x = \lambda_n$, while the second gives a step-function which is left-continuous at $x = \lambda_n$, and for $k > 0$ both the definitions lead to the same Riesz sum $A^k(x)$. Now Theorem 1.81 has two parts, 1.81 (a) and 1.81 (b), and so does Theorem 1.82. The reader will realize that *some* conclusions which are not only correct but also 'significant' do emerge from Theorems 1.81 (a), 1.82 (a) if we choose definition (2), and from 1.81 (b), 1.82 (b) if we choose definition (1). Let us illustrate this by considering the following well-known Tauberian results:

THEOREM P. *If* $\lambda_n a_n = O(\lambda_n - \lambda_{n-1})$, *and* Σa_n *is summable* (λ, k), $k > 0$, *then* Σa_n *converges.*

THEOREM Q. *If* $\lambda_n a_n = O(\lambda_{n+1} - \lambda_n)$, *and* Σa_n *is summable* (λ, k), $k > 0$, *then* Σa_n *converges.*

The hypothesis of Theorem P implies that

$$A(\lambda_n + t) - A(\lambda_n) = O(t\,\lambda_n^{-1}),\ 0 < t = O(\lambda_n),$$

so that Theorem P becomes a corollary to Theorem 1.82 (b). Similarly in Theorem Q, the hypothesis implies that

$$A'(\lambda_n) - A'(\lambda_n - t) = O(t\,\lambda_n^{-1}),\ 0 < t \leqslant \delta\,\lambda_n,\ 0 < \delta < 1,$$

so that Theorem Q becomes a corollary to Theorem 1.82 (a).

From this point of view, it will be clear that each of the two versions given of each of Theorems 1.81 and 1.82 is relevant and significant. It will be observed, however, that we have sought to apply only the (b)-versions of Theorems 1.81 and 1.82 in the sequel.

3. In the third place, we wish to point out here that it is possible to give several versions of the hypotheses on $A(x)$ in Theorem 1.81 without altering either the conclusion or the proof. The resulting versions will vary in their generality and their elegance. Thus in Theorem 1.81 (a), for example, any one of the following four hypotheses on $A(x)$ is sufficient to arrive at the stated conclusion. If we write

$$\vartheta(x) = \{W/V\}^{1/(k+\gamma)},$$

we may have

(i) $A(x) - A(x-t) = O[t^{\gamma} V(x)]$, $0 < t = O(\vartheta(x))$, $\gamma > 0$, (already in the text).

(ii) For any positive-valued function $F(x)$, $x > 0$, such that $F(x) = O(\vartheta(x))$,

$$A(x) - A(x-t) = O[\{F(x)\}^{\gamma} V(x)],\ 0 < t \leqslant F(x),\ \gamma > 0.$$

(iii) There exists a positive-valued function $\phi(\eta)$, defined for $\eta > 0$, such that $\phi(+0) = 0$, and such that

$$A(x) - A(x-t) = O[\phi(\eta)\{\vartheta(x)\}^{\gamma} V(x)],$$

for $0 < t < \eta\, \vartheta(x)$, $\eta > 0$, $\gamma \geqslant 0$.

(iv) There exists a sequence $\{\eta_n\}$, such that $\eta_n > 0$ and $\eta_n \to 0$ as $n \to \infty$, and a function $\phi(\eta)$, positive for positive η, such that $\phi(\eta_n) \to 0$, as $\eta_n \to 0$, and

$$A(x) - A(x-t) = O[\phi(\eta_n)\{\vartheta(x)\}^{\gamma} V(x)],$$

for $0 < t < \eta_n \vartheta(x)$, $\gamma \geqslant 0$.

All these four versions of the hypothesis on $A(x)$ are valid in Theorems 1.81 (a), 1.81 (b), 1.82 (a), 1.82 (b) and 1.83. [It may be noted that Theorem 1.83 is again really two results combined into one, namely, (a) that (1.85) implies (1.88), and (b) that (1.86) implies (1.89). Its significance lies in its use in the proof of Theorem 1 84 where, however, one works with $A^1(x)$ instead of $A(x)$]. Apparently (iv) is the most general version, but we preferred (i) for its elegance. Incidentally it may be noted that in versions (i) and (ii) of Theorems 1.81, 1.82 and 1.83 we could have $\gamma = 0$ provided that in the conclusion about $A(x)$, 'O' replaces 'o'.

At this stage it may also be pointed out that in Theorem 1.81 (a), if we have $t > x$, then with the convention that $A(x) \equiv 0$ for $x \leqslant 0$, the hypothesis reduces to $A(x) = O(t^{\gamma} V)$, and the analysis still holds. Again, in the enunciation of Theorem 1.84, the words "defined for $x > 0$" may be deleted, since $x - K\zeta$ and x' may be negative in some cases.

There is yet a third way of defining the partial sum which is adopted, for instance, in Poisson's summation formula. We meet

with it in Chapter III, pp. 80-81. If we agree to call it $A''(x)$, then $A''(x) = \frac{1}{2}[A(x) + A'(x)]$, so that $A''(\lambda_n) = a_0 + \dots + a_{n-1} + \frac{1}{2} a_n$. If $\lambda_n < x < \lambda_{n+1}$, we have $A(x) = A'(x) = A''(x)$, and the higher Riesz sums are the same in all three cases. With this new definition, however, only versions (ii), (iii) and (iv) of Theorem 1.81 will be significant.

Version (iii) has a connexion with the proof of Theorem 4.53 on which Kuttner has made some remarks [2, p.88]. In the first place, his remark that (4.54) "is clearly false, for its truth for arbitrarily small t would imply that $S(x)$ was continuous" is 'clearly' erroneous. In fact, (4.54) only implies that, for a given R,

$$S[(R+t)^{1/2}] - S(R^{1/2}) = O(t\, R^{k/2-a-1}),$$

for $0 < t = O(R)$. Here, by definition (p.111), we have

$$S(R^{1/2}) = \sum_{r=0}^{n} A_r(x),\ n \leqslant R < n+1,$$

where x is a fixed point in k-space, so that S *is* right-continuous (on the analogy of the partial sum A defined earlier in this note). The truth of (4.54) for arbitrarily small positive t only implies that S is right-continuous at R, and Kuttner's remark is incorrect.

That (4.54) is actually true, at least for $k \geqslant 5$, can be seen easily, not by an appeal to Lemma 4.52, but by using the formula

$$R_l(x) = \pi^{k/2} \{\Gamma(\tfrac{1}{2}k+1)\}^{-1/2} x^{k/2} + O(x^{k/2-1}),$$

which is given in the same paper by Walfisz from which Lemma 4 52 itself was quoted by us without proof. [*Math. Zeit.* 19(1924), 302].

To prove Theorem 4.53 for all dimensions k, it is sufficient to observe that

$$\int_R^{R+t} \frac{dR_k(x)}{x^a} = O(T R^{k/2-a-1}).\ 0 < t \leqslant T = \eta R,\ \eta > 0.$$

In other words, in equation (4.54), replace t by T, where $0 < t < T = \eta R, \eta > 0$. Now the result follows by an appeal to Theorem 1.81 (b), version (ii), stated above. Theorem 4.53 is therefore correct.

A similar situation occurs on p.26, line 13, where '$O_L\ (t\,\omega^{-1})$' should be either deleted or replaced by '$O_L\,(T\,\omega^{-1}), 0 < t \leqslant T = \delta\,\omega$.'

We take this opportunity of remarking that an adaptation of the proof of Theorem 1.83 will help to prove in a simple way the following variant of Theorem 1.89 [cf. S. Minakshisundaram and C.T. Rajagopal, *Proc. London Math. Soc.* 50 (1948), 254 ; Cor. 1A)].

THEOREM R. *If* $\lambda_n a_n = O_L(\lambda_n - \lambda_{n-1})$, *and* $\lambda_n a_n = O_L(\lambda_{n+1} - \lambda_n)$, *then* $\Sigma\, a_n$ *is convergent, if it is summable* (λ, k), $k > 0$.

We assume, as usual, that the 'sum' is zero. If $\lambda_n a_n = O_L\ (\lambda - \lambda_{n-1})$, then

$$A\,(\lambda_n + t) - A\,(\lambda_n) = O_L\,(t\,\lambda_n^{-1}), \tag{3}$$

for $0 < t \leqslant \delta\,\lambda_n$, $\delta > 0$; and if $\lambda_n a_n = O_L\,(\lambda_{n+1} - \lambda_n)$, then

$$A'(\lambda_n) - A'(\lambda_n - t) = O_L\,(t\,\lambda_n^{-1}), \tag{4}$$

for $0 < t \leqslant \delta\,\lambda_n$, $0 < \delta < 1$. If we now use (3) and (4), and argue as in (1.88) and (1.89), we obtain $\lim\limits_{n\to\infty} A_n = 0$.

4. The brief descriptive account of the long technicalities that ensue in Chapter II contains (p.35) the statement :

(H_3). *If* $1 \prec \mu(\lambda) \prec \lambda^{\delta}$, *for every* $\delta > 0$, *then the method* $(\mu,\ k)$ *is more powerful than* (λ, k).

In this statement the phrase 'more powerful than' was not defined by us, and has to be interpreted in a sense appropriate to the context. It was shown, very early in the book, in Corollary 1.62, on page 13 of Chapter I, that if λ_n increases 'much too rapidly', then (λ, k) sums only convergent series. If λ_n and μ_n both increase so rapidly that they reduce to convergence, there will be no point in (H_3), or for that matter in Kuttner's statement of Theorem B [2, p.86]. This trivial possibility can be covered by interpreting 'more powerful than' in the 'wide' sense. There is no question of the correctness of (H_3).

Turning now to Theorem 2.41, the proof consists in showing that [1, pp. 41-44]

$$I_1 + I_2 = o[\{\phi(x)\}^k] + o[\{x\phi'(x)\psi(x)\}^k].$$

This is done by showing that

$$I_2 = o[\{x\phi'(x).\psi(x)\}^k],$$

and $I_1 = I_{1,1} + I_{1,2}$, where

$$I_{1,1} = o[\{x\phi'(x)\psi(x)\}^k],$$

and

$$I_{1,2} = o[\{\phi(x)\}^k]. \qquad [(2.59),\ \text{p.}43].$$

The proof of the orders of I_2 and $I_{1,1}$ has been given by us in full. To establish the order of $I_{1,2}$, defined in (2.53), we have said (cf. p.43) that

$$I_{1,2} = o\left[\int_0^{x_1} t^{k-p-1}\{\phi(x) - \phi(t)\}^p \{\phi'(t)\}^{k-p} dt\right] \tag{5}$$

$$= \begin{cases} o\left[\{\phi(x)\}^p \int_0^{x_1} t^{k-p-1}\{\phi'(t)\}^{k-p} dt\right], \ (p \geqslant 0) & (6) \\ o\left[\{x\phi'(x)\}^p \int_0^{x_1} t^{k-p-1}\{\phi'(t)\}^{k-p} dt\right], \ (p < 0) & (7) \end{cases}$$

$$= o\left[\{\phi(x)\}^p \int_0^{x_1} \{\phi(t)\}^{k-p-1} \phi'(t)\, dt\right] \tag{8}$$

$$= o[\{\phi(x)\}^k]. \tag{9}$$

We should have '$p \geqslant 0$' in (8). The proof of (9), in case $p < 0$, follows directly from (7) *if ϕ' is non-decreasing*, since $k - p \geqslant 1$ and $x\phi'(x) = O\{\phi(x)\}$.

If $p < 0$, and *ϕ' is decreasing*, (9) can be proved directly from (5). We note, first of all, that if $p < 0$, then k is non-integral, and $p = k - h - 1 - r$, where h is the greatest integer less than k, as in the book, and r is an integer such that $0 \leqslant r \leqslant l - h$, $l > k$. We can show that (9) is true for each of these values of p.

If $p = k - h - 1$, then $p + 1 > 0$, and we write (5) as

$$I_{1,2} = o\left[\int_0^{x_1} \{\phi(x) - \phi(t)\}^p \phi'(t) . t^{k-p-1} \{\phi'(t)\}^{k-p-1} dt\right]$$

$$= o\left[\{\phi(x)\}^{k-p-1} \int_0^{x_1} \{\phi(x) - \phi(t)\}^p \phi'(t)\, dt\right]$$

$$= o[\{\phi(x)\}^k].$$

since $p + 1 > 0$.

If $p = k - h - 2$, then $p + 2 > 0$, while $p + 1 < 0$. Integrating the right-hand side of (5) by parts, we get

$$C_1 \{\phi(x) - \phi(x_1)\}^{p+1} x_1^{k-p-1} \{\phi'(x_1)\}^{k-p-1} +$$

$$+ C_2 \int_0^{x_1} \{\phi(x) - \phi(t)\}^{p+1} \phi'(t) . t^{k-p-1} \{\phi'(t)\}^{k-p-3} \phi''(t)\, dt +$$

$$+ C_3 \int_0^{x_1} \{\phi(x) - \phi(t)\}^{p+1} \phi'(t) . t^{k-p-2} \{\phi'(t)\}^{k-p-2} dt.$$

The integrated term is

$$O[\{(x - x_1)\phi'(\eta)\}^{p+1} \{x_1 \phi'(x_1)\}^{k-p-1}], \quad x_1 < \eta < x$$

$$= O[\{\phi(x)\}^k],$$

if we note the definition of x_1 on p.41, and that $p + 1 < 0$, and that ϕ' is monotone decreasing, and use Lemma 2.33 together with the fact that $t\,\phi'(t) = O\{\phi(t)\}$. The estimates of the other two integrals are similar since, by hypothesis, $t\,\phi''(t) = O\{\phi'(t)\}$. And since $k - p - 2 \geqslant 0$, we again use the fact that $t\,\phi'(t) = O\{\phi(t)\}$, and argue as before to establish (9).

If $p = k - h - r - 1$, then we have only to integrate the right-hand side of (5), by parts, r times to get the result.

Thus Theorem 2.41 is correct.

5. The following misprints have come to our notice: On p.21, (1.83), A' should be A. On p.25, line 21, a_r should be a_m. On p.42, (2.54), $\{x^r \psi(x)\}^k$ should be $x^r \{\psi(x)\}^k$. On p.44, in (2.59)6, $\psi(x\}$

should be $\psi(x)\}$. On p.81, line 1, read $h(s)\ e^{\lambda_{m+1}s}$ for $h(s)$. On p.89, line 20, 3.73 should be 3.71. On p.107, last line, 238 should be 246.

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A NOTE ON SYMPLECTIC COMPLEMENTS

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1. Let $J_m = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}$, E being the unit matrix of order m. The symplectic group $Sp(m)$ is defined to be the set of real matrices M such that

$$M'JM = J, \tag{1}$$

M' being the transposed of M. $Sp(m)$ contains a discrete subgroup $M(m)$, called the modular group of degree m, consisting of integral matrices M satisfying (1). A symmetric matrix S of $2m$ rows is called abelian if

$$SJS = \lambda J, \tag{2}$$

$\lambda \neq 0$ is a real number called the kernel of S. In a recent paper we studied the reduction theory of abelian matrices under the modular group $M(m)$.

If we wish to extend Siegel's theory of quadratic forms to abelian forms, it seems necessary to study the theory of complements associated with representations. Let S and T be two rational abelian matrices of $2m$ and $2n$ rows respectively $m \geqslant n$. Let an integral matrix X satisfy

$$X'SX = T. \tag{3}$$

(3) is said to be a *symplectic representation* of T by S if

$$X'J_m X = \lambda J_n, \qquad \lambda \neq 0. \tag{4}$$

It is said to be a *primitive representation* if $\lambda = 1$. A matrix $Y = Y^{(2m,\, 2m-2n)}$ is said to be a *complement* of a primitive X if

$$\left.\begin{aligned} Y' J_m Y &= J_{m-n} \\ M = (C\,F\,D\,G) &\text{ is modular} \end{aligned}\right\}, \tag{5}$$

where $X = (C\ D)$, $Y = (F\ G)$, C, D are matrices of $2m$ rows and n columns and F, G matrices of $2m$ rows and $m-n$ columns. Our object in this note is to prove that complements of a primitive X exist, and to obtain a parametric representation for all complements of a primitive X. Our theorems are analogous to Siegel's theorems on complements associated with primitive representations of quadratic forms. In the end we make a few remarks on the modular group in an algebraic number field.

2. Unless mentioned otherwise all matrices are integral. We use the notation and terminology of Siegel's.

A pair of matrices $C = C^{(m,r)}$ and $D = D^{(m,r)}$, $r \leqslant m$ is said to form a *symmetric* (m, r) *pair* if $X = \begin{bmatrix} C \\ D \end{bmatrix}$ and

$$X' J_m X = 0, \tag{6}$$

and X is of rank r. It will be called a *coprime* (m, r) *pair* if X is primitive. Two symmetric (m,r) pairs $\begin{bmatrix} C \\ D \end{bmatrix}$ and $\begin{bmatrix} F \\ G \end{bmatrix}$ are said to be *conjunctive* if there is a modular matrix M such that

$$M \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}.$$

Conjunctive pairs thus form a class. We shall now prove

THEOREM 1. *Any two coprime* (m, r) *pairs are conjunctive.*

The proof depends on the following

LEMMA 1. (Siegel). *Any two coprime* $(m, 1)$ *pairs are conjunctive.*

PROOF. Let $X = \begin{bmatrix} C \\ D \end{bmatrix}$, where $C' = (c_1, \dots, c_m)$, $D' = (d_1, \dots, d_m)$. It is enough to prove that X is conjunctive with a column $\begin{bmatrix} P \\ A \end{bmatrix}$, where $P' = (1, 0, \dots 0)$. For if $Q' = (e_1, \dots e_m)$,

$$T = \begin{bmatrix} e_1, \dots e_m \\ \vdots \\ e_m \quad\quad 0 \end{bmatrix},$$

and M the modular matrix $\begin{bmatrix} E & O \\ -T & E \end{bmatrix}$, then $M \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

So let U be a unimodular matrix, such that $UD = \begin{bmatrix} d \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where d is the g. c. d. of $d_1,\dots,d_m$. Let $U'^{-1}C = \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}$. Since $\begin{bmatrix} U'^{-1} & 0 \\ 0 & U \end{bmatrix}$ is modular, X is conjunctive with $\begin{bmatrix} P \\ Q \end{bmatrix}$, where

$$P = \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \text{ and } Q = \begin{bmatrix} d \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If V is the unimodular matrix of $(m-1)$ rows such that $U \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} = \begin{bmatrix} e \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then $M = \begin{bmatrix} 1 & 0 & & O \\ 0 & V & & \\ & & 1 & 0 \\ & O & 0 & V'^{-1} \end{bmatrix}$ is modular, and

$M \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix}$, where $P_1 = \begin{bmatrix} e_1 \\ e \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $Q_1 = \begin{bmatrix} d \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $(e_1, e, d) = 1$.

Let now $\lambda = 1$ if all the primes dividing e divide e_1, or d, otherwise λ equals the product of primes dividing e but not e_1, or d. Put $P = \begin{bmatrix} \lambda & 0 \\ 0 & O \end{bmatrix}$ and M_2 equal to the modular matrix $\begin{bmatrix} E & P \\ O & E \end{bmatrix}$. Then $M_1 \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, $L_1' = (e_1 + \lambda d, e, 0,\dots 0)$. Let now W be the unimodular matrix such that $WL_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. This exists because by the

above choice of λ, $(e_1 + \lambda d, e) = 1$. If M_2 equals the modular matrix $\begin{bmatrix} U & O \\ O & U'^{-1} \end{bmatrix}$ then $M_2 \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ has the property required and our lemma is proved.

Let us now prove the theorem.

Let $X = \begin{bmatrix} C \\ D \end{bmatrix}$. Since the first column of X is a coprime $(m, 1)$ pair, it follows that X is conjunctive with $X_1 = \begin{bmatrix} C_1 \\ D_1 \end{bmatrix}$, where

$$C_1 = \begin{bmatrix} 1 & L' \\ 0 & C_2 \end{bmatrix}, D_1 = \begin{bmatrix} O & N' \\ O & D_2 \end{bmatrix}. \tag{7}$$

L and N are column vectors of $r - 1$ rows. X_1 is also a coprime (m, r) pair so $N = 0$ and $\begin{bmatrix} C_2 \\ D_2 \end{bmatrix}$ is a coprime $(m-1, r-1)$ pair. By applying Lemma 1 successively we find X conjunctive with a pair $\begin{bmatrix} C_0 \\ D_0 \end{bmatrix}$ where

$$C_0 = \begin{bmatrix} 1 & & & & * \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & O & & \cdot & \\ & & & & 1 \end{bmatrix} \text{ and } D_0 = O. \tag{8}$$

But this obviously is conjunctive with $\begin{bmatrix} E_r \\ O \\ O \end{bmatrix}$, which proves the theorem.

Theorem 1 shows us that any coprime (m, r) pair can be completed into a modular matrix. If M_1 and M_2 are two such completions of X, then $M_2^{-1} M_1$ is a modular matrix whose first r columns are $\begin{bmatrix} E_r \\ O \\ O \end{bmatrix}$. Thus the modular matrices which are completions of a given coprime (m, r) pair constitute a complete left coset of $M(m)$ by means of the subgroup Δ_r of the modular matrices whose first r columns are $\begin{bmatrix} E \\ O \\ O \end{bmatrix}$. Let us find the matrices in Δ_r.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in Δ_r. Then

$$A = \begin{bmatrix} E & A_2 \\ O & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, C = \begin{bmatrix} 0 & C_2 \\ 0 & C_4 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}, \quad (9)$$

E, B_1, D_1 and O are square matrices of order r.

It follows then that $\begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}$ is a modular matrix of $2m-2r$ rows.

Put $M_1 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$, where

$$A_0 = \begin{bmatrix} E & 0 \\ 0 & A_4 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}, C_0 = \begin{bmatrix} 0 & 0 \\ 0 & C_4 \end{bmatrix}, D_0 = \begin{bmatrix} E & 0 \\ 0 & D_4 \end{bmatrix}, \quad (10)$$

then $M\,M_1^{-1}$ is also in Δ_r and if $M\,M_1^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then

$$P = \begin{bmatrix} E & L \\ 0 & E \end{bmatrix}, \quad Q = \begin{bmatrix} W & G \\ G' & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ -L' & E \end{bmatrix}, R = 0 \text{ and } S = P'^{-1}. \quad (11)$$

L is a matrix of r rows and $m-r$ columns and W is symmetric of order r. We have thus a parametric representation for all modular matrices with given first r columns.

3. We shall now be concerned with pairs of coprime pairs. We prove

THEOREM 2. *Let* $X = \begin{bmatrix} C \\ D \end{bmatrix}$ *a coprime* (m, r) *pair and* $Y = \begin{bmatrix} F \\ G \end{bmatrix}$ *a coprime* (m, s) *pair,* $r \leqslant s \leqslant m$. *Then there exists a modular matrix* $M = (X\,P\,Y\,Q)$ *if and only if* $X'JY = \begin{bmatrix} E & O \end{bmatrix}$, $E = E_r$.

PROOF. That the condition is necessary is obvious since $M'JM = J$. We shall prove it is sufficient. It is enough to prove the theorem for two pairs X_1, Y_1, where $X_1 = MX$, $Y_1 = MY$, for a modular M. Let us choose M in such a way that

$$Y_1 = \begin{bmatrix} L \\ T \end{bmatrix}, L = 0, T = \begin{bmatrix} E_r & 0 \\ 0 & E_{s-r} \\ 0 & 0 \end{bmatrix}. \quad (12)$$

Since $X_1' J Y_1 = \begin{bmatrix} E & O \end{bmatrix}$, it follows that

$$X = \begin{bmatrix} N \\ R \end{bmatrix}, N = \begin{bmatrix} E_r \\ 0 \\ C_3 \end{bmatrix}, R = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}. \tag{13}$$

D_1, D_2 are matrices of r columns but of r and $s-r$ rows respectively. Furthermore since X is a coprime pair $D_1 + C'_3 D_3$ is symmetric. Put now

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 0 \\ E_{s-n} & 0 \\ 0 & E_{m-s} \end{bmatrix}, P_2 = \begin{bmatrix} D_2 & D_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{14}$$

and

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, Q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, Q_2 = \begin{bmatrix} -C_3' \\ 0 \\ E_{m-s} \end{bmatrix}. \tag{15}$$

Then $(X_1 \; P \; Y_1 \; Q)$ is a modular matrix. This proves our theorem.

We remark that the condition $r \leqslant s$ is not restrictive. If $r > s$ then $X' J Y = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $E = E_r$. To prove that even now the theorem is true, observe that

$$(-Y)' J X = \begin{bmatrix} E & O \end{bmatrix}, \tag{16}$$

so by Theorem 2 there exists a modular matrix $M = (-Y_* \; X_*)$. Consider MJ^{-1}. Then

$$MJ^{-1} = (X_* \; Y_*). \tag{17}$$

Let us now find all modular matrices of the form $(X \; P \; Y \; Q)$ with given X and Y. If M_1, M_2 are two such matrices then $M_1^{-1} M_2$ is a modular matrix of the form

$$\begin{bmatrix} E_r & & & \\ 0 & * & O & * \\ & & E_3 & \\ O & * & 0 & * \end{bmatrix}, \tag{18}$$

so that all the modular matrices of the form $(X * Y *)$ form a complete left coset of the full modular group in regard to the subgroup

of modular matrices of the form (18). Call this subgroup $\Delta_{r,s}(m)$. We shall investigate the nature of these groups.

Let $M = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$, where

$$P = \begin{bmatrix} E_r & * & * \\ O & P_3 & P_4 \\ O & P_5 & P_6 \end{bmatrix}, R = \begin{bmatrix} O & O & * \\ O & O & R_2 \\ O & O & R_3 \end{bmatrix},$$
$$Q = \begin{bmatrix} O & * & * \\ O & Q_3 & Q_4 \\ O & Q_5 & Q_6 \end{bmatrix}, \quad S = \begin{bmatrix} E_r & * & * \\ O & E_{s-r} & S_2 \\ O & O & S_3 \end{bmatrix}, \tag{19}$$

be a matrix in $\Delta_{r,s}(m)$. Then the matrix

$$M_1 = \begin{bmatrix} P_3 & P_4 & O & R_2 \\ P_5 & P_6 & O & R_3 \\ Q_3 & Q_4 & E & S_2 \\ Q_5 & Q_6 & O & S_3 \end{bmatrix} \tag{20}$$

is a modular matrix of $2(m-r)$ rows. If we put $M_2 = \begin{bmatrix} P_0 & R_0 \\ Q_0 & S_0 \end{bmatrix}$, where

$$P_0 = \begin{bmatrix} E & 0 & 0 \\ 0 & P_3 & P_4 \\ 0 & P_5 & P_6 \end{bmatrix}, R_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & R_2 \\ 0 & 0 & R_3 \end{bmatrix},$$
$$Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_3 & Q_4 \\ 0 & Q_5 & Q_6 \end{bmatrix}, \quad S_0 = \begin{bmatrix} E & 0 & 0 \\ 0 & E & S_2 \\ 0 & 0 & S_3 \end{bmatrix}, \tag{21}$$

then $M_2 \in \Delta_{r,s}(m)$ and a little calculation shows that $M M_2^{-1}$ is the identity matrix. So any matrix $M \in \Delta_{r,s}(m)$ is of the form (21). If we consider the matrix $\begin{bmatrix} P_6 & R_3 \\ Q_6 & S_3 \end{bmatrix}$ then this is modular of order $2(m-s)$. Put $M_3 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

$$A = \begin{bmatrix} E_{s-r} & 0 \\ 0 & P_6 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & R_3 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & Q_6 \end{bmatrix}, D = \begin{bmatrix} E & 0 \\ 0 & S_3 \end{bmatrix} \tag{22}$$

then, a little calculation shows that

$$M_1 M_3^{-1} = \begin{bmatrix} E & 0 & 0 & 0 \\ P_5 & E & 0 & 0 \\ Q_3 & Q'_5 & E & -P'_5 \\ Q_5 & 0 & 0 & E \end{bmatrix}. \tag{23}$$

Thus the

THEOREM 3. *Every matrix M in $\Delta_{r,s}(m)$ is of the form*

$$\begin{bmatrix} U' & O \\ U^{-1}S & U^{-1} \end{bmatrix} M_1, \tag{24}$$

where $U' = \begin{bmatrix} E_r & 0 & 0 \\ 0 & E_{s-r} & 0 \\ 0 & L & E \end{bmatrix}$ *is unimodular and* $S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W & N \\ 0 & N' & 0 \end{bmatrix}$,

L, N' being arbitrary matrices of $(m-s)$ rows and $(s-r)$ columns and W an $(s-r)$-rowed symmetric matrix, and $M_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *where* $A = \begin{bmatrix} E_s & 0 \\ 0 & A_1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & C_1 \end{bmatrix}$, $D = \begin{bmatrix} E_s & 0 \\ 0 & D_1 \end{bmatrix}$ *and* $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ *is a symplectic matrix of $2(m-s)$ rows.*

COROLLARY. *The group $\Delta_{r,r}(m)$ is isomorphic to the modular group of degree $(m-r)$.*

Let $M = (X\ P\ Y\ Q) \in \Delta_{r,s}(m)$. Then

$$J M J^{-1} = \begin{bmatrix} \begin{matrix} E_s \\ 0 \end{matrix} & * & O & * \\ O & * & \begin{matrix} E_r \\ 0 \end{matrix} & * \end{bmatrix}$$

showing that $JMJ^{-1} = M'^{-1} \in \Delta_{s,r}(m)$. This shows

COROLLORY. *$\Delta_{r,s}(m)$ and $\Delta_{s,r}(m)$ are conjugate subgroups of the full modular group and the mapping $M \to M'^{-1}$ takes one onto the other.*

From the definition of these groups, it follows

$$\Delta_{1,s}(m) \supset \Delta_{2,s}(m) \supset \ldots \supset \Delta_{s,s}(m) \supset \Delta_{s,s+1}(m) \supset \ldots \supset \Delta_{r,m}. \tag{25}$$

Consider the groups $\Delta_{r,s}(m)$ and $\Delta_{s,r}(m)$. If M is a matrix in $\Delta_{r,s}(m) \cap \Delta_{s,r}(m)$ then M'^{-1} is also in this intersection. Since $\Delta_{s,s}(m)$, if $s \geqslant r$ is already contained in this intersection by (25), it follows from Theorem 3 that

COROLLARY. $\Delta_{s,s}(m) = \Delta_{r,s}(m) \cap \Delta_{s,r}(m), \quad r \leqslant s.$

4. Let us now consider the corresponding situation in algebraic number fields.

Let K be an algebraic number field of finite degree over the field k of rationals and σ be an automorphism of K such that σ^2 is the identity automorphism. Let K_1 be the fixed field of σ. If σ is not the identity then K/K_1 has degree 2. Let for any matrix $M = (a_{ij})$ with integral elements in K, M^* denote the matrix

$$M^* = (b_{ij}), \; b_{ij} = a_{ji}^{\sigma}. \tag{26}$$

Then the mapping $M \to M^*$ is an involution in the ring of m by m integral matrices over K. The *modular group of K of degree m* is defined as the group of matrices M with

$$M^* J M = J. \tag{27}$$

The involution is said to be of the first kind if σ is identity otherwise of the second kind. If we wish to prove analogues of Theorem 3 it is essential to prove

LEMMA 2. *If X is a column vector of $2m$ rows of integers of K which are coprime then there is a modular matrix M, such that*

$$MX = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We shall prove it in case of involutions of the first kind. In case of involutions of second kind a similar procedure is possible.

Let $X = \begin{bmatrix} C \\ D \end{bmatrix}$, $C' = (c_1, \ldots, c_m)$, $D' = (d_1, \ldots, d_m)$ and $(c_1, \cdots, d_m)$

$= 1$. Since every ideal is g.c.d. of two integers there is a unimodular matrix U such that $UD = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ 0 \end{bmatrix}$. Since $\begin{bmatrix} U'^{-1} & O \\ O & U \end{bmatrix}$ is modular and MX also has the same property as M we may assume even in the beginning that X has it. Let $g = (c_3, \ldots, c_m)$. Let S be the symmetric matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & \lambda & O \\ 0 & \lambda & 0 & \\ & O & & O \end{bmatrix}, \tag{28}$$

and M_1 the modular matrix $\begin{bmatrix} E & S \\ O & E \end{bmatrix}$ then M_1X has first m elements as c_1, c_2, $c_3 + \lambda d_2$, $c_4, \ldots, c_m$. Let A_1 be the set of prime ideals dividing c_4 but not c_3, B_1 the set dividing c_4, c_3 but not d_2, C_1 the set dividing c_4, c_3, d_2 and dividing c_3 to the same power as d_2. These sets are mutually disjoint. So choose λ so that it is a multiple of ideals in A_1, B_1 but prime to those in c_1. Then $(c_3 + \lambda d_2, c_4, \ldots, c_m) \mid d_2$. We may therefore assume $g \mid d_2$ even in the beginning. Thus $(c_1, c_2, g, d_1) = 1$.

Let T be the symmetric matrix $\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ of order m and M_3 the modular matrix $\begin{bmatrix} E & T \\ O & E \end{bmatrix}$. M_3X has the first m elements as $c_1 + \mu d_1$, $c_2, c_3 \ldots$. Let A denote set of prime ideals dividing g and c_2 but not c_1 or d_1, B the set of ideals dividing g, c_2 and c_1, and C those which divide g, c_2 and d_1. These sets are mutually disjoint. Choose μ to be a multiple of ideals in A but prime to those in B and C. Then MX has first m elements coprime. So we may assume $(c_1, \ldots, c_m) = 1$. If U_1 is unimodular, so that $U_1 C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then

$$\begin{bmatrix} U_1 & O \\ O & U'^{-1}_1 \end{bmatrix} X = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ e_1 \end{bmatrix}$$

We now complete the proof by the same method as in Lemma 1.

In the case of involutions of the second kind a further reduction is necessary by way of choosing c_2 in such a way that if a prime ideal p divides c_2, the conjugate p' (over K_1) if distinct from p does not divide c_2 In this way one arrives at the

THEOREM 4. *If M is modular then $|M| = \epsilon^\sigma \epsilon^{-1}$, where ϵ is a unit in K.*

Theorem 3 can be proved in a similar way. Theorems 1 and 2 are not in general true. However, if X and Y are primitive matrices satisfying (6), then Theorems 1 and 2 are still true.

Let λ be an integer in K, in case of involutions of the second kind, such that $\lambda^\sigma = -\lambda$. Then the modular group over K is the same as the group of units over K of the hermitian form $X^* h X$ where $h = \lambda \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}$. It can be proved that the units of a hermitian form over K or more generally the units of fixed points in involutorial algebras have a finite number of generators. We shall return to this problem later.

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ON A CLASS OF LINEAR SPACES IN FUNCTION THEORY

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1. The object of this note is to complete a theorem of V. Ganapathy Iyer [2] on the space of integral functions, and to uphold it for the more general spaces defined by O. Toeplitz and G. Köthe [3]. We recall the definitions and results from [3].

2. Preliminaries. π_r is the space of all power-series $x(\omega) = \sum_{n=1}^{\infty} x_n \omega^{n-1}$ with radius of convergence R, $0 < r \leqslant R \leqslant \infty$; so that π_∞ is the space of entire functions.

ρ_r is the space of all power-series $u(z) = \sum_{n=1}^{\infty} u_n z^{n-1}$, with radius of convergence R, $0 < r < R \leqslant \infty$.

ρ_0 is the space of all power-series with radius of convergence $R > 0$.

To every $x(\omega) \in \pi_r$ [respectively $u(z) \in \rho_r$] we associate in a one-to-one manner the vector $x = (x_1, x_2, \ldots)$ [respectively $u = (u_1, u_2, \ldots)$]. As co-ordinate spaces these spaces are characterized by the following conditions :

$$\pi_\infty: \limsup_{n\to\infty} |x_n|^{1/n} = 0, \quad \pi_r: \limsup_{n\to\infty} |x_n|^{1/n} \leqslant 1/r;$$

$$\rho_r: \limsup_{n\to\infty} |u_n|^{1/n} < 1/r; \quad \rho_0: \limsup_{n\to\infty} |u_n|^{1/n} < \infty.$$

We then define

$$|x| = \sup_n |x_n|^{1/n}, \text{ and } |u| = \sup_n |u_n|^{1/n},$$

Thanks are due to Professor K. Chandrasekharan and Dr. K. G. Ramanathan for their help in the preparation of this note.

and

$$ux = \sum_{n=1}^{\infty} u_n x_n, \text{ whenever the series converges.}$$

It is easily verified that $|x - y|$, for $x, y \in \pi_r$, defines a metric in π_r, and $|u - v|$, for $u, v \in \rho_r$ a metric in ρ_r.

Define a linear space E to be *locally convex* if it is a topological linear space which is Hausdorff and for which there exists a complete system of open *convex* neighbourhoods of 0. A set $A \subset E$ is *convex* if whenever $x, y \in A$, we have $\lambda x + (1-\lambda) y \in A$, $0 \leqslant \lambda \leqslant 1$. The *dual* E' of E is the space of all continuous linear functions on E. E' can be endowed with the structure of a locally convex space, in the usual way. E'' is the dual of E'. There exists a *canonical* mapping of E into E''. E is *reflexive* if this mapping is an isomorphism onto E''. Suppose E, E' are two vector spaces for which a bilinear functional $B(x, x')$ is defined on $E \times E'$, so that (1) $B(x, x') = 0$ for all $x' \in E'$ implies $x = 0$; (2) $B(x, x') = 0$ for all $x \in E$ implies $x' = 0$. Then E is endowed with the structure of a locally convex space for which E' is the dual [1].

Now, ux, defined above, gives us such a bilinear functional on $\pi_r \times \rho_{1/r} (0 < r \leqslant \infty)$, and π_r and $\rho_{1/r}$ are *dual to one another* [3, §1, Satz 1] In π_r, in addition to this locally convex topology, there is defined a metric. The relation between the several topologies in π_r will be studied in another note.

3. The result. Let $x = \sum_{n=1}^{\infty} x_n \, \omega^{n-1}$ and $y = \sum_{n=1}^{\infty} y_n \, \omega^{n-1}$ be elements in π_r. Define $x \circ y = \sum_{n=1}^{\infty} x_n y_n \omega^{n-1}$. $x \circ y$ is again an element of π_r for $r \geqslant 1$. Denote $(x)_n = (x \circ x \circ \ldots \circ x)$ n-times. Let $L[x]$ denote the closed linear space generated by $(x)_n$ for $n = 1, 2, \ldots$ in the locally convex topology.

V. G. Iyer has proved, for the space π_∞, the following result: *If* $x(\omega) \in \pi_\infty$, *where* $x(\omega) = \sum_{n=1}^{\infty} x_n \omega^{n-1}$, *with* (i) $x_i \neq x_j$ for $i \neq j$, *and* (ii) $|x_i| > 0$, *then* $L[x] = \pi_\infty$ [2, Theorems A and B].

We shall now prove the following

THEOREM. *In order that $L[x] = \pi_r$, $x \in \pi_r$ and $r > 1$, it is necessary and sufficient that*

$$x_i \neq x_j \text{ for } i \neq j \text{ and } |x_i| > 0 \text{ for } i \geqslant 1. \qquad (*)$$

We remark that this theorem includes V. G. Iyer's result.

For the proof we use the following result due to Dieudonné [1]. Let E and E' be two linear spaces in duality, and A be a closed subspace of E. Denote by A^* the set of $u \in E'$ with $\langle x, u \rangle = u(x) = 0$, for every $x \in A$. Let A' be the dual of A. Then with weak topologies on A', E' and A^*, we have $A' = E'/A^*$. $A = E$ if and only if $A^* = \{0\}$.

The condition is necessary. For let $L[x] = \pi_r$. Then we can apply the above result on weak topology. We have $L[x]^* = \{0\}$. So, if $u \in \rho_{1/r}$, $u \neq 0$, then $u \in L[x]^*$. In particular take $u_i = z^i$, and $u_{i,j} = z^i - z^j$. Then $u_i \neq 0$, $u_{i,j} \neq 0$ for every i and every $i \neq j$, i.e. they are non-trivial forms on π_r Suppose $x_i = x_j$. Then $x_i^n = x_j^n$ for $n \geqslant 1$. Also $u_i(x) = x_i$, and $u_i((x)_n) = x_i^n$. So $u_{i,j}((x)_n) = 0$ for $n \geqslant 1$. Hence $u_{i,j}(y) = 0$, for every $y \in L[x]$. Therefore $u_{i,j} = 0$, a contradiction So $x_i \neq x_j$ for $i \neq j$. Similarly $|x_i| > 0$ for $i \geqslant 1$.

Sufficiency is proved by showing that $L[x]^* = \{0\}$. Let $u \in \rho_{1/r}$ and $u \in L[x]^*$. Then $u((x)_n) = 0$ for $n \geqslant 1$. So $\sum_{p=1}^{\infty} u_p x_p^n = 0$ for $n \geqslant 1$. Moreover for $r > 1, x_p \to 0$, for by taking $\omega = 1$ in $x(\omega)$, $\sum_{p=1}^{\infty} |x_p| < \infty$. These two results show that $g(z) = \sum_{p=1}^{\infty} u_p x_p e^{x_p z}$ is an entire function with $g^{(n)}(0) = 0$, for $n = 0, 1, 2, \ldots$ Hence $g(z) \equiv 0$ Since $x_p \to 0$ and condition (*) holds, we can apply Theorem B[2], and get $u_p x_p = 0$ and $u_p = 0$ for every p. So $u = 0$.

Moreover our method proves that condition (*) of the theorem is necessary even for $r = 1$. That it is not sufficient for $r = 1$, however, is shown by the following.

Take

$$x = \sum_{n=1}^{\infty} n\, \omega^{n-1},\ y = \sum_{n=1}^{\infty} \omega^{n-1}.$$

Then $x, y \in \pi_1$ and x satisfies (*). If the condition were sufficient, we would have a sequence of complex numbers $\{\lambda_1^{(q)}, \ldots, \lambda_{kq}^{(q)}\}$ such that $\sum_{i=1}^{kq} \lambda_i^q(x)_i \to y$ as $q \to \infty$; i.e. for any $u \in \rho_1$, $u\left(\sum_{i=1}^{kq} \lambda_i^q(x)_i\right) \to uy$.

Taking $u = u_N$, we have

$$\left\{\sum_{i=1}^{kq} \lambda_i^q N^i - 1\right\} \to 0 \text{ as } q \to \infty \text{ for } N = 1, 2, \ldots.$$

So for $q \geqslant q_0$, $\left|\sum_{i=1}^{kq} \lambda_i^q N^i - 1\right| < 1$ for $N = 1, 2, \ldots.$

But as $N \to \infty$, $\left|\sum_{i=1}^{kq} \lambda_i^q N^i - 1\right| \to \infty$ for $q = 1, 2, \ldots$ and so for $N > N_0$,

$$\left|\sum_{i=1}^{kq_0} \lambda_i^{q_0} N^i - 1\right| > 1.$$

This gives us a contradiction.

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ALGEBRAIC BASIS OF MORSE'S VARIATIONAL THEORY

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INTRODUCTION. The aim of this work is to bring out clearly the algebraic basis of Morse's theory of critical levels and critical points both in his original formulation [6*a*] and in its further generalizations [1, 6*b*, 6*c*, 7].

To motivate the algebraic formulation we have in view, we shall briefly recapitulate the classical example of a twice continuously differentiable function f defined on a compact n-dimensional C^3-manifold M^n with isolated singular points (i.e. points $x \in M^n$, at which $\frac{\partial f}{\partial x_i} = 0$, $i = 1, \ldots, n$). Since M^n is compact, there are a finite number of critical points $m_1, \ldots, m_p$, and consequently a finite number of critical levels $c_1, \ldots, c_p$, where $c_i = f(m_i)$.

Let a_i, b_i, $i = 1, \ldots, p$, be reals such that $a_{i+1} = b_i$, $a_i < c_i < b_i$, and there is no critical level in (a_i, b_i) other than c_i.

In this situation, the following facts can be proved.

(1). $H(f \leqslant a_i) = H(f < c_i)$, where $H = \Sigma H^k$ being the kth homology group, and

$$f \leqslant a_i = \{x \in M^n \,|\, f(x) \leqslant a_i\},$$

while

$$f < c_i = \{x \in M^n \,|\, f(x) < c_i\}.$$

(2). $H(f \leqslant c_i) = H(f \leqslant b_i)$.

Both these follow from various deformations which are available because of the existence of the orthogonal trajectories to the level surfaces [6*a*, 7].

This work was carried out under the supervision of Professor K. Chandrasekharan to whom the author is indebted for constant encouragement throughout.

(3). $H(f \leqslant c_i, f < c_i) = H(f \leqslant b_i, f \leqslant a_i)$.

This follows at once from (1) and (2).

(4). Let m_i^k be the rank of $H^k(f \leqslant b_i, f \leqslant a_i)$ and $M^k = \Sigma\, m_i^k$. Let R^k be the kth Betti number of M^n. Morse's relations can be considered to be

$$\sum_{r=0}^{k} (-1)^r M^{k-r} \geqslant \sum_{r=0}^{k} (-1)^r R^{k-r}, \quad k = 0, \dots, n,$$

equality holding for $k = n$, and these can be obtained purely algebraically by a theorem on exact sequences [4].

Originally Morse's relations were expressed in terms of the ranks of the critical groups at the level c_i, i.e. $H(f \leqslant c_i, f < c_i)$, a level c_i being called *topologically critical* if the rank of the corresponding group does not vanish. It will be observed that on reading (1)—(4) backwards, one gets the suggestion that in the quite general situation where M^n is replaced by a topological space, one may proceed as in (4) in order to secure an algebraic basis of Morse's theory, provided, of course, one is sure that one can 'localize' the groups occurring in (4) at the level c_i (i.e. (3) should hold). This localization can be effected by means of certain axioms on f or by the prescription of certain deformations analogous to those which arise in (1) and (2) [5, 6*b*, 6*c*, 6*d*, 7]. Such a theory however would be limited to the case in which the critical levels are isolated; further, it would require axiomatization of facts which are not algebraic.

We therefore use a more general definition of critical levels which, on the one hand, allows us to deal with the case when neither the critical levels nor the critical points are isolated, and on the other allows us to deduce Morse's relations without any special restrictions on the given topological space, or on the given function, or on the homology theory. We define a critical level as follows : c is a critical level if $\operatorname{dir}\lim_{\epsilon > 0} \{H(f \leqslant c, f \leqslant c - \epsilon)\}$ is non-trivial, and using this definition we shall obtain in Theorem 3.4 certain exact sequences, which we shall call generalized Morse relations, from which, in meaningful cases, we shall deduce the actual Morse relations for any

function f and any topological space X In fact we carry this out for any abelian group with a differential operator on which a filtration is defined, thus bringing out clearly the algebraic basis of the theory. We first develop in Part I, comprising §§ 1, 2, an axiomatic theory of critical levels in which we describe various exact sequences arising out of the data of a homology theory with an indexing set. Then in Part II we apply this to an abelian group with a differential operator and a filtration. In part III we apply the results of I and II to the case of a function on a space X.

It will be noticed that we have not so far given any topological definition of critical points. To assertions (1)—(4) in the example considered above, one may add the following assertion about critical points :

(5). If $x_{i_1}, \ldots, x_{i_t}$ are critical points at the level c, then

$$H(f \leqslant c, f < c) = H(f < c + x_{i_1} + \ldots + x_{i_t}, f < c),$$

and

$$R^k(f \leqslant c, f < c) = \sum_{m=1}^{t} R^k(f < c + x_{i_m}, f < c).$$

This also can be carried over to the general case but we shall do this elsewhere.

The author's interest in this subject was stimulated by the work of Morse and specially by [1, 7] in which the idea of considering groups with a filtration, and of defining critical levels by direct limits, is implicit. The recent notes of R. Deheuvels, based as they are on the work of J. Leray, contain many interesting results which effect a far-reaching generalization of Morse's relations by algebraic means. The present paper is influenced by R. Deheuvels's results and ideas ; in particular, the definition of critical levels which is used here, though it differs from his definition in as much as it uses direct limits instead of his inverse limits, is strongly suggested by his work. It will be seen, however, that this difference in definition leads to more general results.

The axiomatic theory which is given in Part I of this paper, and which was developed on the suggestion of Professor S. Eilenberg,

gives a generalization of Morse's relations, in any homology theory and on any topological space. This generalization can be effected in more ways than one, either by direct or by inverse limits. Both these procedures are discussed here. It turns out that all the relations formulated in (4) can be obtained always by using direct limits, but even with restrictions on the homology one can obtain only partial results (the so-called second set of Morse's relations $M^k \geqslant R^k$) if one uses inverse limits. It must be acknowledged that those results on inverse limits which form part of this paper, in the case of groups with a differential operator and a filtration, and of topological spaces, were first given by R. Deheuvels [1]. But some of them are here extended, while some others have been carried over to the general axiomatic set-up outlined above.

The author wishes to express his sincere thanks to Professor S. Eilenberg for stimulating discussions and helpful suggestions regarding the earlier part of this paper. Thanks are also due to Mr. M. S. Narasimhan who made many helpful comments on the exposition of the text.

I. Axiomatic theory.

1. Homology theory with indexing set

1.1. Let R be a partially ordered set, i.e. a set with a binary relation '$\geqslant$' which is reflexive and transitive. We write (p, q) if $p \geqslant q$, (p, q, r) if $p \geqslant q \geqslant r$, and so on. Also we write $(p, q) \geqslant (p', q')$ if $p \geqslant p'$, $q \geqslant q'$, and so on.

Definition 1.1. *A homology theory with R as an indexing set consists of the following objects :*

(*i*) *for each (p, q) an abelian group $H(p, q)$ is given ;*

(*ii*) *for $(p, q) \geqslant (p', q')$ a homomorphism $\pi^{p,q}_{p',q'} : H(p', q') \to H(p, q)$ is given ; and*

(*iii*) *for (p, q, r) a homomorphism $d : H(p, q) \to H(q, r)$ is given.*

These terms (i), (ii), (iii) are subject to the following four axioms.

H$_1$. *$H(p, q) \to H(p, q)$ is the identity homomorphism.*

H$_2$. *If $(p, q) \geqslant (p', q') \geqslant (p'', q'')$, then the following diagram*

$$\begin{array}{ccc} H(p'', q'') & \to & H(p, q) \\ & \searrow \quad \nearrow & \\ & H(p', q') & \end{array}$$

is commutative.

H$_3$. *If $(p, q, r) \geqslant (p', q', r')$, then the following diagram*

$$\begin{array}{ccc} H(p', q') & \xrightarrow{d} & H(q', r') \\ \downarrow & & \downarrow \\ H(p, q) & \xrightarrow{d} & H(q, r) \end{array}$$

is commutative.

H$_4$. *For every (p, q, r) the relations $(p, q) \geqslant (p, r) \geqslant (q, r)$ give rise to the following exact sequence :*

$$\ldots \to H(q, r) \to H(p, r) \to H(p, q) \xrightarrow{d} H(q, r) \to \ldots$$

We obtain, at once, $H(p, p) = 0$. We need only consider the exact sequence corresponding to the triple (p, p, q) :

$$\ldots \to H(p, q) \to H(p, q) \to H(p, q) \xrightarrow{d} H(p, q) \to \ldots.$$

If R has a maximum element we denote it by $+\infty$, and if it has a minimum element we denote it by $-\infty$. In the latter case we write $H(p, -\infty)$ as $H(p)$ simply. In this case, instead of giving axioms for triples, we could have given these for couples, as follows. In (iii) we put $r = -\infty$ and in H$_3$, H$_4$ put $r = -\infty$. The homomorphism d for the triples would then be defined by

$$d : H(p, q) \xrightarrow{d} H(q) \to H(q, r).$$

1.2. Let R^4 be the set of quadruples (p, q, r, s), where $p, q, r, s \in R$, and $p \geqslant q \geqslant r \geqslant s$.

R^4 is ordered by the relation : $(p, q, r, s) \geqslant (p', q', r', s')$ if and only if $p \geqslant p', q \geqslant q', r \geqslant r', s \geqslant s'$.

Definition 1.2.1. *Let*

$$E(p, q, r, s) = im\left\{H(q, s) \rightarrow H(p, r)\right\},$$

and

$$D(p, q, r, s) = \text{im}\left\{\begin{array}{ccc} H(p, r) & \rightarrow & H(q, s) \\ & \searrow d \quad \nearrow & \\ & H(r, s) & \end{array}\right\},$$

where the maps are defined by (ii) and (iii) of Definition 1.1.1.

If $(p, q, r, s) \geqslant (p', q', r', s')$, we have homomorphisms

$$\pi^{p,q,r,s}_{p',q',r',s'} : E(p', q', r', s') \rightarrow E(p, q, r, s),$$

and

$$\pi^{p,q,r,s}_{p',q',r',s'} : D(p', q', r', s') \rightarrow D(p, q, r, s),$$

defined by the following diagrams :

$$\begin{array}{ccc} H(q', s') & \rightarrow & H(p', r') \\ \downarrow & & \downarrow \\ H(q, s) & \rightarrow & H(p, r) \end{array} \qquad\qquad \begin{array}{ccc} H(p', r') & \xrightarrow{d} & H(q', s') \\ \downarrow & & \downarrow \\ H(p, r) & \xrightarrow{d} & H(q, s) \end{array}$$

where the map is defined as in Definition 1.2.1.

Theorem 1.2.2. *For each* (p, q, r, s, t) *the following five sequences are exact :*

(i) $0 \rightarrow E\,(p, q, r, t) \rightarrow E\,(p, q, r, s) \rightarrow D\,(q, r, s, t) \rightarrow 0;$

(ii) $0 \rightarrow D\,(p, q, r, s) \rightarrow E\,(q, r, s, t) \rightarrow E\,(p, r, s, t) \rightarrow 0;$

(iii) $0 \rightarrow E\,(p, r, s, t) \rightarrow E\,(p, q, s, t) \rightarrow E\,(p, q, r, t) \rightarrow 0;$

(iv) $0 \rightarrow D\,(p, q, s, t) \rightarrow D\,(p, q, r, t) \rightarrow D\,(p, q, r, s) \rightarrow 0;$

(v) $0 \rightarrow D\,(q, r, s, t) \rightarrow D\,(p, s, r, t) \rightarrow D\,(p, q, s, t) \rightarrow 0.$

The maps are as detailed in the above paragraph.

Remark. These sequences tell us what happens when one of the co-ordinates of the E's or the D's is changed, and exhaust all possibilities. In short, we can say that for the E's a change in the first

or third co-ordinate induces an endomorphism, while a change in the second or fourth induces a monomorphism. For the D's the case is exactly the reverse. For the proof we require a lemma.

Lemma. *Consider the commutative diagram*

$$\begin{array}{ccccc} & & B & & \\ & \theta_2 \nearrow & \downarrow \theta_1 & & \\ A' & \xrightarrow{\theta_3} & A & \xrightarrow{\psi_3} & A'' \\ & & \downarrow \psi_1 & \psi_2 \nearrow & \\ & & C & & \end{array}$$

in which the horizontal row is exact. Then the following sequence is exact :

$$0 \to \operatorname{im}(A' \to C) \to \operatorname{im}(B \to C) \to \operatorname{im}(B \to A'') \to 0.$$

Or, more precisely,

$$0 \to \operatorname{im}(\psi_1\ \theta_3) \to \operatorname{im}(\psi_1\ \theta_1) \to \operatorname{im}(\psi_3\ \theta_1) \to 0$$

is exact.

Proof. Exactness at the ends follows from the factorization $\psi_1\theta_3 = \psi_1\theta_1\theta_2$ and $\psi_3\theta_1 = \psi_2\psi_1\theta_1$. For the middle term, we have

$$\begin{aligned} \ker\,[\operatorname{im}\,(\psi_1\theta_1) \to \operatorname{im}\,(\psi_3\theta_1)] &= \ker\,\theta_2 \cap \operatorname{im}\,\psi_1\theta_1 \\ &= \psi_1\,(\operatorname{im}\,\theta_1 \cap \ker\,(\psi_2\psi_1)) \\ &= \psi_1\,(\operatorname{im}\,\theta_1 \cap \ker\,\psi_3) \\ &= \psi_1\,(\operatorname{im}\,\theta_1 \cap \operatorname{im}\,\theta_3), \\ &\qquad \text{by exactness of the middle row,} \\ &= \psi_1\,(\operatorname{im}\,\theta_3) = \operatorname{im}\,(\psi_1\theta_3) \\ &= \operatorname{im}\,[\operatorname{im}\,(\psi_1\,\theta_3) \to \operatorname{im}\,(\psi_1\theta_1)]. \end{aligned}$$

The theorem follows on applying this lemma to the five diagrams given below with H omitted throughout.

$$\begin{array}{ccccc} & & (q, s) & & \\ & \nearrow & \downarrow & & \\ (q, t) & \rightarrow & (q, r) & \rightarrow & (r, t) \\ & & \downarrow & d \nearrow & \\ & & (p, r) & & \end{array} \tag{1}$$

$$\begin{array}{ccccc} & & (r, t) & & \\ & d \nearrow & \downarrow & & \\ (p, r) & \xrightarrow{d} & (r, s) & \rightarrow & (p, s) \\ & & \downarrow & \nearrow & \\ & & (q, s) & & \end{array} \tag{2}$$

$$\begin{array}{ccccc} & & (q, t) & & \\ & \nearrow & \downarrow & & \\ (r, t) & \rightarrow & (p, t) & \rightarrow & (p, r) \\ & & \downarrow & \nearrow & \\ & & (p, s) & & \end{array} \tag{3}$$

$$\begin{array}{ccccc} & & (p, r) & & \\ & \nearrow & \downarrow & & \\ (p, s) & \rightarrow & (p, q) & \xrightarrow{d} & (q, s) \\ & & \downarrow d & \nearrow & \\ & & (q, t) & & \end{array} \tag{4}$$

$$\begin{array}{ccccc} & & (p, s) & & \\ & & \downarrow d & & \\ (q, s) & \xrightarrow{d} & (s, t) & \rightarrow & (q, t) \\ & & \downarrow & \nearrow & \\ & & (r, t) & & \end{array} \tag{5}$$

2. Extensions of the homology theory.

2.1. Extension of the indexing set. Let O be an ordered set consisting of two elements $(0, 1)$, $1 > 0$. We introduce an ordering relation in $R' = R \times O$ in the following way : $(a, x) \geqslant (b, y)$ if and only if $a > b$, or $a \geqslant b$ and $x \geqslant y$.

Let α, β be the following maps :

$$\alpha : R \rightarrow R' : \alpha\, a = (a, 1)$$

$$\beta : R' \rightarrow R : \beta(a, x) = a.$$

The two maps are order-preserving, and α is one-one. We identify the image by α of R' in R with R, and we write the remaining elements, viz. $(a, 0)$ as $a - 0$. We have: $a > a - 0$ and $a > b$ implies $a > a - 0 > b$.

2.2. EXTENSION OF THE HOMOLOGY GROUPS. Let R be now a set directed at each level, i.e. the set $[p \in R \mid q \geqslant p]$ is directed for each q. It is easily seen that R' is also directed at each level.

Suppose a homology theory with R as indexing set is given. We define a homology theory with R' as indexing set as follows.

DEFINITION 2.2.1. *For each* $\omega_1, \omega_2 \in R'$, *let* $[\omega_1, \omega_2]$ *be the directed set formed by* $\{(p, q)\}$ *such that* $(p, q) \in R \times R$, *and* $(\omega_1, \omega_2) \geqslant (p, q)$. *We define* $H(\omega_1, \omega_2) = \operatorname{dir\,lim} \{H(p, q), \pi^{p, q}_{p', q'}\}$, *where the limit is taken over the directed set* $[\omega_1, \omega_2]$, *and* $\pi^{p, q}_{p', q'}$ *is the homomorphism* $: H(p', q') \to H(p, q)$, $(p, q) \geqslant (p', q')$.

Next we define a homomorphism $\pi^{\omega_1, \omega_2}_{\omega_1', \omega_2'}: H(\omega_1', \omega_2') \to H(\omega_1, \omega_2)$, where $(\omega_1, \omega_2) \geqslant (\omega_1', \omega_2')$. Towards that end we need to define a map of directed sets, and for each map the corresponding homomorphisms. The map of directed sets is given by $\alpha: [\omega_1', \omega_2'] \to [\omega_1, \omega_2]$, $\alpha(p, q) = (p, q)$, the corresponding homomorphisms by $\pi^{\alpha(p, q)}_{p, q}: H(p, q) \to H(\alpha(p, q)) = H(p, q)$.

Further for $(\omega_1, \omega_2, \omega_3)$ we have a map $d: H[\omega_1, \omega_2] \to H(\omega_2, \omega_3)$ as follows. For $(\omega_1, \omega_2, \omega_3)$ consider the directed set $\{(p_1, p_2, p_3)\}$ with $p_1 \geqslant p_2 \geqslant p_3$ and $\omega_i \geqslant p_i$, $i = 1, 2, 3$. For each (p_1, p_2, p_3) we have $d: H(p_1, p_2) \to (p_2, p_3)$.

This induces a homomorphism of direct limits.

$$d: H(\omega_1, \omega_2) \to H(\omega_2, \omega_3).$$

A priori direct limits are here taken over the directed set $(\omega_1, \omega_2, \omega_3)$ and d is thus a homomorphism between $\operatorname{dir\,lim}_{[\omega_1, \omega_2, \omega_3]} H(p_1, p_2)$ and $\operatorname{dir\,lim}_{[\omega_1, \omega_2, \omega_3]} H(p_2, p_3)$. But since $\alpha: [\omega_1, \omega_2, \omega_3] \to [\omega_1, \omega_2]$ given

by $\alpha(p_1, p_2, p_3) = (p_1, p_2)$ is an allowed map, and its image is the whole of (ω_1, ω_2), and the corresponding group homomorphisms are the identity isomorphisms by [2, p. 223], it follows that $\operatorname{dir\,lim}_{[\omega_1, \omega_3, \omega_3]} H(p_1, p_2) = \operatorname{dir\,lim}_{[\omega_1, \omega_2]} H(p_1, p_2)$, and thus we obtain the required homomorphism.

NOTE. We observe that if, in particular, $(\omega_1, \omega_2) \in R \times R$, then $H(\omega_1, \omega_2)$ as defined in Definition 2.2.1 is isomorphic to $H(\omega_1, \omega_2)$ as given earlier. This will show that it makes sense to call $\{H(\omega_1, \omega_2)\}$ an extension of $\{H(p, q)\}$ as the former coincides on elements of $R \times R$ with $\{H(p, q)\}$. For, firstly in such a case the directed set $[\omega_1, \omega_2]$ has a maximum element (ω_1, ω_2), the set consisting of (ω_1, ω_2) alone is cofinal in $[\omega_1, \omega_2]$, and secondly the map $\alpha : [\omega_1, \omega_2] \to [\omega_1, \omega_2]$ and the corresponding homomorphism define a map of direct systems $\underset{[\omega_1, \omega_2]}{H(\omega_1, \omega_2)}$ to $H(\omega_1, \omega_2)$, which satisfies the conditions of [2, p. 223] thus giving $\operatorname{dir\,lim}_{[\omega_1, \omega_2]} H(\omega_1, w_2) = H(\omega_1, \omega_2)$.

THEOREM 2.2.2. *The system formed by the groups $H(\omega_1, \omega_2)$, maps $\pi^{\omega_1, \omega_2}_{\omega_1', \omega_2'}$ and $d(\omega_1, \omega_2, \omega_3)$ is a homology theory with R' as indexing set.*

PROOF. Axioms H_1, H_2, H_3 follow trivially. For every $(p_1, p_2, p_3) \in [\omega_1, \omega_2, \omega_3]$ we have an exact sequence

$$\cdots \to H(p_2, p_3) \to H(p_1, p_3) \to H(p_1, p_2) \to H(p_2, p_3) \to \cdots.$$

Since direct limits preserve exact sequences, taking direct limits over $[\omega_1, \omega_2, \omega_3]$, we have

$$\cdots \to H'(\omega_2, \omega_3) \to H'(\omega_1, \omega_3) \to H'(\omega_1, \omega_2) \to H'(\omega_2, \omega_3) \to \cdots$$

But as shown in the last paragraph of Definition 2.2.1, $H'(\omega_2, \omega_3) \approx H(\omega_2, \omega_3)$ and so on. This gives the exact sequence

$$\cdots \to H(\omega_2, \omega_3) \to H(\omega_1, \omega_3) \to H(\omega_1, \omega_2) \to H(\omega_2, \omega_3) \to \cdots$$

as required.

DEFINITION 2.2.3. *We denote by $H(\omega_1, \omega_2)$ the homology theory proved to exist in Theorem 2.2.2, and call it an extension of the given homology theory with R as indexing set.*

As we did in 1.2 we can now define

$$E(\omega_1, \omega_2, \omega_3, \omega_4) = \text{im}\,\{H(\omega_2, \omega_4) \to H(\omega_1, \omega_3)\},$$

$$D(\omega_1, \omega_2, \omega_3, \omega_4) = \text{im}\left\{\begin{array}{ccc} \{H(\omega_1, \omega_3) & \to & H(\omega_2, \omega_4)\} \\ d\searrow & & \nearrow \\ & H(\omega_3, \omega_4) & \end{array}\right\}.$$

For these groups then hold the exact sequences of 1.2.

We can characterize these alternatively as follows :

THEOREM 2.2.4. We have

$$E(\omega_1, \omega_2, \omega_3, \omega_4) = \text{dir lim}\,\{E(p,q,r,s), \pi^{p,q,r,s}_{p',q',r',s'}\}$$

direct limits being taken over the directed set $[\omega_1, \omega_2, \omega_3, \omega_4]$.

PROOF. dir lim $\{E(p, q, r, s), \pi^{p,q,r,s}_{p',q',r',s'}\}$

$$= \text{dir lim im}\,\{H(q, s) \to H(p, r)\}$$

$$= \text{im}\,\{\text{dir lim } H(q, s) \to \text{dir lim } H(p, r)\}$$

(because direct limits commute with im)

$$= \text{im}\,\{H(\omega_2, \omega_4) \to H(\omega_1, \omega_3)\}$$

$$= E(\omega_1, \omega_2, \omega_3, \omega_4).$$

2.3. INVERSE EXTENSION OF THE HOMOLOGY THEORY WITH R AS INDEXING SET.

DEFINITION 2.3.1. *A partially exact homology theory with R as indexing set is one defined as in* 1.1 *except that* H_4 *need not hold.*

Given a homology theory with R as indexing set, we obtain a partially exact homology theory as follows. Let $[\omega_1, \omega_2]$ be the directed set formed by $\{(p,q)\}$ with $(p,q) \in R \times R$ and $(p,q) > (\omega_1,\omega_2)$. We define

$$H(\omega_1, \omega_2) = \text{inv lim}\,\{H(p,q), \pi^{p,q}_{p',q'}\},$$

where the limits are taken over the directed set $[\omega_1, \omega_2]$ and $\pi^{p,q}_{p',q'}$ are the homomorphisms $H(p', q') \to H(p, q)$. For $(\omega_1, \omega_2) > (\omega_1', \omega_2')$ the map α of directed sets $\alpha: [\omega_1, \omega_2] \to [\omega_1', \omega_2']$ obtained by inclusion, and the corresponding homomorphisms of homology

groups, give rise to a homomorphism $\pi^{\omega_1,\ \omega_2}_{\omega_1',\ \omega_2'}: H(\omega_1', \omega_2') \to H(\omega_1, \omega_2)$. For $(\omega_1, \omega_2, \omega_3)$ we have a map $d : H(\omega_1, \omega_2) \to H(\omega_2, \omega_3)$ defined by the system of homomorphisms

$$d : H(p_1, p_2) \to H(p_2, p_3), \qquad \text{as in 2.2.}$$

THEOREM 2.3.2. *The system obtained from* $H(\omega_1, \omega_2)$, $\pi^{\omega_1,\ \omega_2}_{\omega_1',\ \omega_2'}$, $d(\omega_1, \omega_2, \omega_3)$ *is a partially exact homology theory with* R *as indexing set.*

PROOF. We require to verify H_1, H_2, H_3 which is trivial in virtue of the known properties of inverse limits.

Since the above homology theory is not exact, we do not define $E(\omega_1, \omega_2, \omega_3, \omega_4)$ as in Definition 1.2.1 but adopt the alternative definition given in 2.2.4. We note that the two definitions are not equivalent as we cannot interchange inverse limits with images.

DEFINITION 2.3.3. *We have*

$$E(\omega_1, \omega_2, \omega_3, \omega_4) = \text{inv lim } \{E(p, q, r, s),\ \pi^{p,\ q,\ r,\ s}_{p',\ q',\ r',\ s'}\},$$

inverse limits being taken over the directed sets $[\omega_1, \omega_2, \omega_3, \omega_4]$, $\pi^{p,\ q,\ r,\ s}_{p',\ q',\ r',\ s'}$ *being the homomorphism* $E(p', q', r', s') \to E(p, q, r, s)$.

Similarly $D(\omega_1, \omega_2, \omega_3, \omega_4) = \text{inv lim } \{D(p, q, r, s),\ \pi^{p,\ q,\ r,\ s}_{p',\ q',\ r',\ s'}\}$.

If $(\omega_1, \omega_2, \omega_3, \omega_4) > (\omega'_1, \omega'_2, \omega'_3, \omega'_4)$, we have the homomorphisms

$$E(\omega'_1, \omega'_2, \omega'_3, \omega'_4) \to E(\omega_1, \omega_2, \omega_3, \omega_4),$$
$$D(\omega'_1, \omega'_2, \omega'_3, \omega'_4) \to D(\omega_1, \omega_2, \omega_3, \omega_4),$$

obtained by taking the inclusion mapping of directed sets and the corresponding identity homomorphisms.

Since inverse limits do not preserve exact sequences either, we cannot give any theorem similar to Theorem 1.2.2. But we can state the weaker

THEOREM 2.3.4. *The following sequences are exact:*

(1) $0 \to D(\omega_1, \omega_2, \omega_3, \omega_4) \to E(\omega_1, \omega_3, \omega_4, \omega_5) \to E(\omega_2, \omega_3, \omega_4, \omega_5)$,

(2) $0 \to E(\omega_1, \omega_2, \omega_3, \omega_4) \to E(\omega_1, \omega_2, \omega_3, \omega_5) \to D(\omega_2, \omega_3, \omega_4, \omega_5)$,

(3) $0 \to E(\omega_1, \omega_3, \omega_4, \omega_5) \to E(\omega_1, \omega_2, \omega_4, \omega_5) \to E(\omega_1, \omega_2, \omega_3, \omega_5)$,

(4) $0 \to D(\omega_2, \omega_3, \omega_4, \omega_5) \to D(\omega_1, \omega_3, \omega_4, \omega_5) \to D(\omega_1, \omega_2, \omega_3, \omega_5)$,

(5) $0 \to D(\omega_1, \omega_2, \omega_4, \omega_5) \to D(\omega_1, \omega_2, \omega_3, \omega_5) \to D(\omega_1, \omega_2, \omega_3, \omega_4)$.

PROOF. The following lemma together with Theorem 1.2.2 furnish the proof.

LEMMA. *If for each $\alpha \in I$, I being a directed set, we have exact sequences* $0 \to G_\alpha \to H_\alpha \to K_\alpha$, *the diagram*

$$\begin{array}{ccccccccc} 0 & \to & G_\alpha & \to & H_\alpha & \to & K_\alpha & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & G_\beta & \to & H_\beta & \to & K_\beta & \to & 0 \end{array},$$

being commutative, and if G, H, K are inverse limits of G_α, H_α, K_α respectively, then the following sequence is exact :

$$0 \to G \to H \to K \to.$$

PROOF. We know that $\ker (G \to H) = \text{inv lim} \{\ker (G_2 \to H_\alpha)\}$. Therefore $\ker (G \to H) = 0$. Also $G_\alpha \to \text{im } G_\alpha$ is an isomorphism. Hence $G \to \text{inv lim im } G_\alpha$ is an isomorphism [2, p. 219]. Thus $\text{im } G = \text{inv lim im } G_\alpha$. But $\text{inv lim im } G_a = \text{inv lim ker } (H_\alpha \to K_\alpha) = \ker (H \to K)$. Therefore $\text{im } G = \ker (H \to K)$.

We now give a sufficient condition for full exactness. For ease of proof we state it in the case in which R is a linearly ordered set.

THEOREM 2.3.5. *If for all p and every q there exists q' such that $E(p, q''', q'', q) = 0$ whenever $q' > q''' > q'' > q$, then all the sequences are exact.*

PROOF : Step 1. Under the above hypothesis we shall prove that the component homomorphisms in inverse limits yielding E's and D's are isomorphisms. Using (1), (2) of Theorem 1.2.2, i.e.

$$0 \to E(p, q''', q', s) \to E(p, q''', q'', s') \to D(q''', q'', s', s) \to 0$$
$$0 \to D(p', p, q''', q'') \to E(p, q''', q'', s) \to E(p', q''', q'', s) \to 0.$$
$$0 \to E(p, q''', q'', s) \to E(p, q''', q'', q),$$

we observe, since the middle term is zero in (1) and (2), that for any p', p and (i) for every q there exists q' such that $D(p', p, q''', q'') = 0$, (ii) and for any s, s' and for every q there exists q' such that $D(q''', q'', s, s') = 0$ whenever $q' > q''' > q'' > q$. (iii) Further it follows, since $E(p', q'', q'', q) = 0$ for $p > p'$, that $E(p' + 0, q''', q'', q) = 0$ under the same conditions as above.

Similar results are true for the D's.

Step 2. Now we prove that the component homomorphisms of each limit are isomorphisms.

(a) $E(p+0, q, r, s)$. We have

$$0 \to D(p''', p'', q, r) \to E(p'', q, r, s) \to E(p''', q, r, s) \to 0.$$

But by (ii) the first term is zero. Hence

$$E(p''', q, r, s) \approx E(p'', q, r, s) \text{ for } p' > p''' > p'' > p,$$

and so $E(p', q, r, s) \approx E(p+0, q, r, s)$.

(b) $E(p, q+0, r, s)$. We have

$$0 \to E(p, q'', r, s) \to E(p, q''', r, s) \to E(p, q''', q'', s) \to 0.$$

But the last one is zero. Hence $E(p, q'', r, s) \approx E(p, q''', r, s)$, so that $E(p, q+0, r, s) \approx E(p, q', r, s)$ and so on. The process is now straightforward in case only one letter is in R'. Next if two letters are so, take for example $E(p+0, q+0, r, s)$. We have

$$0 \to E(p+0, q'', r, s) \to E(p+0, q''', r, s) \to E(p+0, q''', q'', s). \quad (*)$$

But by (iii) the last term is zero. Hence $E(p+0, q''', r, s) \approx E(p+0, q'', r, s)$ on account of the following lemma.

Lemma. 2.3.6. *Let I and J be two directed sets and $G_{\alpha\beta}$ be an inverse system of groups over $I \times J$.* Let

$$G_\beta = \operatorname*{inv\,lim}_I G_{\alpha\beta}, \; H_\alpha = \operatorname*{inv\,lim}_J G_{\alpha,\beta}.$$

Then G_β and H_α are inverse systems over J and I respectively, and

$$\operatorname*{inv\,lim}_{I\times J} G_{\alpha\beta} \approx \operatorname*{inv\,lim}_J G_\beta \approx \operatorname*{inv\,lim}_I H_\alpha.$$

Proof. We first prove that $\{G_\beta\}$ is an inverse system. For $\beta' > \beta$ and any $\alpha \in I$ we have a map $\pi_{\alpha\beta}^{\alpha\beta'}: G_{\alpha\beta'} \to G_{\alpha\beta}$. The maps satisfy commutativity relations as $\{G_{\alpha\beta}\}$ is an inverse system, and we obtain thus a $\pi_{\beta'}^{\beta}: G_{\beta'} \to G_\beta$. Again, on account of commutativity relations in $\{G_{\alpha\beta}\}$ it follows that $\pi_{\beta''}^{\beta} = \pi_{\beta''}^{\beta'} \cdot \pi_{\beta'}^{\beta}$ proving that $\{G_\beta\}$ is an inverse system. Similarly it can be proved that $\{H_\alpha\}$ is an inverse system.

To prove the isomorphisms, let us take the first one. We have a map $f: I \times J \to J$ given by $f(\alpha, \beta) = \beta$ and corresponding to it the homomorphism of G_β to $G_{\alpha\beta}$. This induces a homomorphism of inv lim $G_\beta \to$ inv lim $G_{\alpha\beta}$. We first prove it is onto. Let $g \in$ inv lim $G_{\alpha\beta}$. Then $g = (g_{\alpha\beta})$ with $g_{\alpha\beta} \in G_{\alpha\beta}$. For each β the elements determine an element $g_\beta \in G_\beta$ and the elements $\{g_\beta\}$ define an element $g' \in$ inv lim G_β which by definition is mapped onto g. It remains to prove that the map is one-to-one. Suppose $x \neq y$, $x, y \in$ inv lim G_β. Then for some β, $x_\beta \neq y_\beta$, $x_\beta, y_\beta \in G_\beta$. But then at least for some (α, β), $x_{\alpha\beta} \neq y_{\alpha\beta}$, for otherwise x_β would be equal to y_β and hence the images of x and y have their $\alpha\beta$th co-ordinates different, so that they are different themselves

For all the remaining five combinations as well as for the D's we proceed as above. Again when 3 or 4 letters are in R' then in any sequence like (*), the initial or final term will contain two letters in R' and by (i), (ii), (iii) will vanish, and we shall have the required isomorphisms.

Step 3. Now we prove the final part. We have here the following general situation. G and H are inverse limits of G_α, H_α respectively, π_β^α being isomorphisms, and there exist epimorphisms $h_\alpha : G_\alpha \to H_\alpha$ for each α. Considering the diagram

$$\begin{array}{ccc} G & \to & H \\ \downarrow & & \downarrow \\ G_\alpha & \to & H_\alpha \end{array},$$

we deduce that the induced natural homomorphism is an epimorphism, which was required to be proved.

Remark. If each $H(p, q)$ is graded with d of order -1 with respect to the degree, i.e. $H(p,q) = \Sigma H^k(p,q)$, then it can be proved that $E(p, q, r, s)$, $D(p, q, r, s)$, $H(\omega_1, \omega_2)$ are all graded. Further if we look at the diagrams of Theorem 1.2 2 it is easily seen that all exact sequences hold with a superscript attached to each member except that in diagram (1). The last one has a superscript $k - 1$ for it is obtained by applying d to the earlier one.

II. Applications to groups with differential operator.

3. It will now be assumed that R is a complete dense-in-itself lattice. We shall denote the group with a differential operator by (A, d).

3.1. Filtrations and decompositions.

Definition 3.1.1. *A decomposition of* (A, d) *according to an indexing set* R *is a function* $f: R \to 2^A$ *such that* (i) $A^p = f(p)$ *is a sub-group* (ii) $p \geqslant q$ *implies* $A^p \supset A^q$, (iii) $dA^p \subset A^p$ *and* (iv) $A = \bigcup_{p \in R} A^p$. *We shall denote such a decomposition by* $\{A^p\}$.

Definition 3.1.2. *A filtration of* (A, d) *according to an indexing set* R *is a function* $f: A \to R$ *such that* (i) $f(a-b)) \leqslant \sup\{f(a), f(b)\}$ *and* (ii) $f(da) \leqslant f(a)$.

Theorem 3.1.3. *If* $\{A^p\}$ *is a decomposition of* (A, d), *then the function* f *defined by* $f: A \to R: f(a) = \inf\{p \mid a \in A^p\}$ *is a filtration of* (A, d). *Conversely if* $f: A \to R$ *is a filtration* (A, d), *then the system* $\{A^p\}$, *defined by the requirement that* $a \in A^p$ *if and only if* $f(a) < p$, *is a decomposition of* (A, d).

Proof. Since, whenever $a \in A^p, b \in A^q$ we have $a - b \in A^{\sup(p,q)}$ it follows that $f(a-b) \leqslant \sup(p, q)$. Hence

$$f(a-b) \leqslant \sup(\min p, \min q)$$
$$\text{for } p, q, \text{ such that } a \in A^p,\ b \in A^q,$$
$$\leqslant \sup(f(a), f(b)).$$

Since $dA^p \subset A^p$ we have $f(da) \leqslant f(a)$. This proves that f is a filtration. To prove the converse we first prove that A^p is a subgroup. Let $a \in A^p$, $b \in A^p$. Then $f(a) < p$ and $f(b) < p$, hence $f(a-b) < p$ and $a - b \in A^p$. Next if $p \geqslant q$ and $a \in A^q$ then $p \geqslant q > f(a)$. Hence $a \in A^p$, so that $A^p > A^q$. Finally $f(da) \geqslant f(a)$ implies $dA^p \subset A^p$.

The filtration and the decomposition, related as in Theorem 3.1.3, are said to be 'associated'.

DEFINITION 3.1.4. *If* $\{A^p\}$ *is a decomposition of* (A, d), *let*

$$A^{p-0} \equiv \bigcup_{p>q} A^q, \quad \text{and} \quad A^{p+0} \equiv \bigcap_{q>p} A^q.$$

A^{p+0} will be called the 'completed group at the level p'.

The group $G_1A = \sum_p A^p/A^{p-0}$ will be called 'the direct group associated with $\{A^p\}$'. The group $G_2A = \sum_p A^{p+0}/A^p$ will be called 'the inverse group associated with $\{A^p\}$'. We observe, if R is a finite set $(p_1, \ldots, p_k)$, and is linearly ordered, that $A^{p_k-0} = A^{p_{k-1}}$ and $A^{p_k+0} = A^{p_{k+1}}$.

3.2. It can be easily proved that if B is an allowed subgroup of $\{A, d\}$, and if f and $\{A^p\}$ are a filtration and a decomposition of $\{A, d\}$, then f restricted to B and $\{B^p = A^p \cap B\}$ are a filtration and a decomposition of B respectively, and further the latter two are associated if the former ones are so. Also $B^{p+0} = A^{p+0} \cap B$.

Let C be the kernel of d and D be its image. We use the notations :

$$\begin{array}{ll} C^p = \{A^p \cap C\} & D^p = \{A^p \cap D\} \\ C^q_s = \{A^q \cap d^{-1}(A^s)\} & D^q_s = \{A^s \cap dA^q\} = dC^q_s \\ C^{p-0} = \{A^{p-0} \cap C\} & D^{p-0} = \{A^{p-0} \cap D\} \\ C^{p+0} = \{A^{p+0} \cap C\} & D^{p+0} = \{A^{p+0} \cap D\}. \end{array}$$

DEFINITION 3.2.1. Let $\{A^p\}$ be a decomposition of (A, d). Then we define (i) for each (p, q) a group $H(p, q) = H(A^p/A^q)$, (ii) for each $(p, q) > (p', q')$ a homomorphism $\pi^{p,q}_{p',q'} : H(p', q') \to H(p, q)$ given by $H(A^{p'}/A^{q'}) \to H(A^p/A^q)$, and (iii) for each (p, q, r) a homomorphism $d : H(p, q) \to H(q, r)$ given by $H(A^p/A^q) \xrightarrow{d} H(A^q/A^r)$ the homology groups being calculated with respect to the differential operator induced by d, which is possible as all subgroups A^p are allowed.

THEOREM 3.2.2. *The system* $\{H(p, q), \pi^{p,q}_{p',q'}, d\}$ *defined above forms a homology theory with* R *as indexing set.*

PROOF. The four axioms follow in virtue of the properties of homology groups of groups with differential operator.

THEOREM 3.2.3.

(i) $H(A^p, A^q) = \dfrac{A^p \cap d^{-1}A^q}{dA^p + A^q} = \dfrac{C^q_p}{dA^p + A^q}$.

(ii) $E(p, q, r, s) \approx \dfrac{C^q_s}{C^r_s + D^q_p}$.

(iii) $D(p, q, r, s) \approx \dfrac{D^r_p}{D^s_p + D^r_q}$.

PROOF.

(i) is well known.

$$\text{(ii)}\quad E(p, q, r, s) = \operatorname{im}\{H(q, s) \xrightarrow{i} H(p, r)\}$$

$$= \frac{H(q, s)}{\ker i}$$

$$= \frac{C^q_s/(dA^q + A^s)}{(A^q \cap d^{-1}A^s) \cap d^{-1}(dA^p + A^r)/(dA^q + A^s)}$$

$$= \frac{C^q_s}{C^r_s + D^q_p}.$$

The proof of (iii) is similar.

3.3. DECOMPOSITIONS OF HOMOLOGY GROUPS OF (A, d) INDUCED BY THOSE OF (A, d).

THEOREM 3.3.1. *Let* $\{A^p\}$ *be a decomposition of* (A, d). *Then*

(i) $\{\bar{A}^{q'}\} = \operatorname{im}\{E(p, q', r, s) \to E(p, q, r, s)\},\ q > q'$,

and

(ii) $\{\bar{A}^{s'}\} = \operatorname{im}\{E(p, q, r, s') \to E(p, q, r, s)\},\ s > s'$,

are decompositions of $E(p, q, r, s)$;

(iii) $\{\bar{A}^{p'}\} = \operatorname{im}\{D(p', q, r, s) \to D(p, q, r, s)\},\ p > p'$,

and

(iv) $\{\bar{A}^{r'}\} = \operatorname{im}\{D(p, q, r', s) \to D(p, q, r, s)\},\ r > r'$,

are decompositions of $D(p, q, r, s)$.

PROOF. If $q'' > q'$, then $\bar{A}^{q''} \supset \bar{A}^{q'}$, and so on from Theorem 1.2.2.

Theorem 3.3.2.

$$\bar{A}^{q'-0} = \operatorname*{dir\,lim}_{q'>q''} \{E(p, q'', r, s)\} = E(p, q'-0, r, s).$$

Proof. We have $0 \to E(p, q'', r, s) \to E(p, q''', r, s)$, $q''' > q''$ exact. Hence the direct limit is $\bigcup_{q'>q''} E(p, q'', r, s) = \bar{A}^{q'-0}$.

Corollary. $G_1(E(p, q, r, s)) = \sum_{q'} E(p, q', q'-0, s)$.

Proof. By definition $G_1(E(p, q, r, s)) = \sum_{q'} \dfrac{\bar{A}^{q'}}{\bar{A}^{q'-0}}$. By the above theorem and the exactness of

$$0 \to E(p, q-0, r, s) \to E(p, q, r, s) \to E(p, q, q-0, s) \to 0,$$

we have

$$\frac{\bar{A}^{q'}}{\bar{A}^{q'-0}} = E(p, q', q'-0, s),$$

from which the corollary follows.

Similar results hold in the case of decompositions (ii), (iii), (iv). Thus we see that the direct summands of G_1 are respectively the E's and D's occurring in the direct extension of the given homology theorey.

Remark. Because Theorem 1.2.2 does not hold for inverse limits, we do not give any results about G_2 $(E(p, q, r, s))$. We shall consider this separately, and consider sufficient conditions so that Theorem 1.2.2 may hold.

3.4. Certain particular homology groups. Generalized Morse inequalities.

Definition 3.4.1. *Let*

$$\begin{aligned}
L_p(h) &= E(p+h, p, p-0, p-h-0), & L(h) &= \Sigma L_p(h);\\
P_p(h) &= E(p+h, p, p-0, -\infty), & P(h) &= \Sigma P_p(h);\\
K_p &= E(\infty, p, p-0, -\infty), & K &= \Sigma K_p;\\
M_p(h) &= D(p, p-0, p-h-0, -\infty), & M(h) &= \Sigma M_p(h);\\
N_p(h) &= D(\infty, p+h, p, p-0), & N(h) &= \Sigma N_p(h);
\end{aligned}$$

when the set R has a group operation; otherwise we make the same definitions omitting h throughout.

NOTE. $L_p(h)$ generalizes the cap groups at level p and of span h defined by M. Morse [6c].

For convenience we shall hereafter omit the 'h' in our writing.

THEOREM 3.4.2. *The following sequences are exact:*

$$0 \to P \to L \to M \to 0,$$
$$0 \to N \to P \to K \to 0.$$

PROOF. Since the direct sum of exact sequences is exact, the result follows from Theorem 1.2.2.

These exact sequences are generalizations of Morse relations. We shall obtain them, in particular, when (i) each L_p, $H(p,q)$ is graded with d of order -1 with respect to the degree of gradation, which will be denoted by a superscript, and (ii) H, L are finitely generated vector spaces and (iii) R is the set of real numbers.

THEOREM 3.4.3. *In the above conditions if l^q is the rank of $L^q(h)$ and R^q is the rank of $H^q(A)$, then*

$$\sum_{i=0}^{k} (-1)^{k-i} (l^i - R^i) \geqslant 0, \quad k = 1,\ldots,n,$$

equality holding for n.

PROOF. Since $L^k = \Sigma_p L^k$ is a finite dimensional vector space, $L_p^k \neq 0$ only at a finite number of values of p say $p_1,\ldots, p_\alpha$. From sequence (i) of Theorem 1.2.2, we have

$$0 \to P_p^k \to L_p^k \to M_p^{k-1} \to 0,$$
$$0 \to N_p^k \to P_p^k \to K_p^k \to 0$$

It follows that K^p also does not vanish at most at those values of p. We then have

$$\begin{aligned} l_{p_i}^k &= m_{p_i}^{k-1} + p_{p_i}^k, \quad i = 1, \ldots, \alpha, \\ p_{p_i}^k &= n_{p_i}^k + k_{p_i}^k, \end{aligned} \tag{1}$$

where k, l, m, n, p are the ranks of the respective groups.

Now consider

$$0 \to E^k(\infty, p-0, -\infty, -\infty) \to E^k(\infty, p, -\infty, -\infty) \to E^k(\infty, p, p-0, -\infty) \to 0.$$

Since the third term does not vanish at most at $p_1, \ldots, p_s$,

$$E(\infty, p_i - 0, -\infty, -\infty) = E(\infty, p_{i-1}, -\infty, -\infty)$$

by the observation following Defn. 3.1.4. Hence

$$e^k_{p_i} = e^k_{p_i} - e^k_{p_i-0}, \; e^k_{p_i-0} = e^k_{p_{i-1}}, \; e^k_{p_1-0} = 0, e^k_{p_s} = R^k, \tag{2}$$

where $e_{p_i}{}^k - e_{p_i-0}{}^k$ are the ranks of the groups $E^k(\infty, p_i, -\infty, -\infty)$ and $E^k(\infty, p_i - 0, \infty, -\infty)$ respectively. Summing (1) over all p_i's and using (2), we have

$$l^k = m^{k-1} + n^k + R^k. \tag{3}$$

Now we show that

$$m^k = n^k. \tag{4}$$

We consider

$$0 \to D^k(p, p-0, q, -\infty) \to D^k(p, p-0, p-h-0, -\infty) = M_p{}^k(h)$$

and

$$0 \to D^k(p, p-0. q-0, -\infty) \to D^k(p, p-0, q, -\infty) \to D^k(p, p-0, q, q-0) \to 0$$

By the same argument as at the beginning of the proof $D^k(p, p-0, q, -\infty)$ vanishes except at a finite number of values of q and consequently we have

$$m_p{}^k = \sum_q m^k_{p,q}$$

where m^k_p is the rank of M^k_p and $m^k_{p,q}$ that of $D^k(p, p-0, q, q-0)$.

Similarly we consider

$$0 \to D^k(q, p+h, p, p-0) \to D^k(\infty, p+h. p, p-0) = N^k_p$$

and

$$0 \to D^k(q-0, \; p+h, p, \; p-0) \to D^k(q, \; p+h, \; p, \; p-0) \to D^k(q, q-0, p, p-0) \to 0.$$

It follows that $n^k{}_p = \Sigma_q n_{q,p}$. But since $m^k{}_{p,q} = n^k{}_{q,p}$, we have

$$\Sigma_p n^k{}_p = \Sigma_{p,q} n^k{}_{p,q} = \Sigma_{p,q} m^k{}_{p,q} = \Sigma_p m^k{}_p,$$

hence $m^k = n^k$. From (3) and (4) we get $\sum_{i=0}^{k} (-1)^{k-i} (l^i - R^i) = n^k$, and since $n^k \geqslant 0$, we have $\sum_{i=0}^{n} (-1)^{k-i} (l^i - R^i) \geqslant 0$.

Corollary. *If all $H^k(p, q) = 0$ for $k > n$, then*

$$\sum_{i=0}^{n} (-1)^{n-i} (l^i - R^i) = 0.$$

Proof. For then $n^n = 0$, and $\sum_{i=0}^{n} (-1)^{n-i} (l^i - R^i) = 0$.

Remark. If we put $h = 0$, we obtain a result which is valid in case R is linearly ordered set.

The direct summands of G_2's of E's and of D's are E's and D's occurring in the inverse extension of homology groups. Since the inverse extension is partially exact we cannot characterize G_2 as we did G_1 in Theorem 3.3.2 nor can we establish exact sequences such as in § 3.4 and Morse's inequalities cannot be stated unrestrictedly. However since homology groups in this case arise out of a filtration on $\{A, d\}$ by suitably restricting f, we may expect to make at least some sequences in the inverse extension exact if not all. Moreover it is seen in 2.3.5 that certain conditions on E's made the inverse extension exact. We give a similar condition in this case. Further if we impose certain conditions on f and on the indexing set R which are fulfilled if R is a set of real n-tuples, though we do not get an exact inverse extension, we do gain those exact sequences which are necessary to establish the generalized Morse inequalities. Many of the results here have been obtained by R. Deheuvels. Notably Theorem 3.5.3 and the definition of accessible filtration are due to him. He announced also the second part of (ii) of Theorem 3.6.2. We complete it to achieve full exact sequences. Towards that end we have to introduce a little more material about filtrations.

Definition 3.5.1. *A decomposition of* $\{A^p\}$ *is said to be attained at* p *if* $A^{p+0} = A^p$.

If we define a new decomposition $\{A'^p\}$ (of A), indexed by R, by setting $A'^p = A^{p+0}$ we easily see that $\{A'^p\}$ is attained everywhere. But if f is a filtration, and $\{A^p\}$ the associated decomposition, then $A^{p+0} = A^p \cup f^{-1}(p)$, therefore $\{A^p\}$ is never attained at values p which are actually taken by f.

If I is a subgroup of A, then we can define naturally two decompositions and a filtration of A/I starting from that of A as follows :

Definition 3.5.2. *If* A^p *is a decomposition of* A, *then*

$$\operatorname{im}(A^p \to A/I) = \frac{A^p + I}{I},$$

and

$$\operatorname{im}(A^{p+0} \to A/I) = \frac{A^{p+0} + I}{I}$$

are decompositions of A/I. *If* f *is a filtration of* A, *then* $\bar{f}$ *defined by* $\bar{f}(\bar{a}) = \inf_{a \in \bar{a}} \{f(a)\}$ *is a filtration of* A/I, *where* $\bar{a}$ *is the image of* a *under* $A \to A/I$.

We shall further notice certain facts which though not directly useful will, to a certain extent, help motivate the ensuing theorems.

(i) There may exist several decompositions yielding the same filtration as for example : if $\{A^p\}$ is the decomposition associated with a filtration, $\{A'^p\}$ as defined above, then $\{A'^p\}$ and $\{A^p\}$ define the same filtration. But given a decomposition, there exists a unique filtration the decomposition associated with which need not agree with the former.

(ii) If we now pass to A/I, we observe that a decomposition of A gives rise to a decomposition of A/I and to a filtration of A. The former yields a filtration of A/I as does the latter. We can prove that both are equal. But if we start with a filtration, and get the

corresponding decompositions of A/I, these need not be the same. This latter fact is further related to the following question:

(iii) If A^p is attained at p, is $\bar{A}^p = \dfrac{A^p + I}{I}$ so attained ?

We have

$$A^{p+0} = \operatorname*{inv\,lim}_{p'>p} A^{p'}, \text{ and } \bar{A}^{p+0} = \operatorname*{inv\,lim}_{p'>p} \{\operatorname{im}(A^{p'} \to A/I)\}.$$

If A^p is attained at p, then $A^{p+0} = A^p$, and if $\bar{A}^p$ is attained at p then $\bar{A}^{p+0} = \bar{A}^p$. That is

$$\operatorname*{inv\,lim}_{p'>p} \{\operatorname{im}(A^{p'} \to A/I)\} = \operatorname{im}\{\operatorname*{inv\,lim}_{p'>p} A^{p'} \to A/I\},$$

so that what we require is that the inverse limit should commute with images, which, in general, is not true. We do give a sufficient condition on the filtration f for this to hold, where $\{A^p\}$ is defined by f. We fix the notation as follows :

$$\{A'^p\} = \{A^{p+0}\}, \{\bar{A}^p\} = \frac{A^p + I}{I}, \bar{A}'^p = \frac{A'^p + I}{I}.$$

THEOREM 3.5.3. *A necessary and sufficient condition that the decomposition $\bar{A}'^p$ be attained is that in each class $\bar{a}$ there should exist h such that $f(\bar{a}) = f(h)$.*

Necessity.—Let $\bar{A}'^{p+0} = \bar{A}'^p$. Let, for any $\bar{a}$, $\bar{f}(\bar{a}) = p$. Then $\bar{a} \in \bar{A}'^{p+0}$ (see lemma below). Hence $\bar{a} \in \bar{A}'^p$. Therefore there exists $h \in \bar{a}$ such that $h \in A'^p = A^{p+0}$. Hence $f(h) \leqslant p = \bar{f}(\bar{a})$. But, by the definition of $\bar{f}(\bar{a})$, $f(h) \geqslant p$, giving $f(h) = p$.

Sufficiency.—Let for any $\bar{a}$ there exist $h \in \bar{a}$ such that $\bar{f}(\bar{a}) = f(h) = p$. Let $\bar{a} \in \bar{A}'^{p+0}$. Then $\bar{a} \in \bar{A}'^{p'}$ for $p' > p$, so that $\bar{f}(\bar{a}) \leqslant p$ From this it follows that there exists $h \in \bar{a}$ such that $f(h) = \bar{f}(\bar{a}) \leqslant p$. Hence $a \in A'^p$ so that $\bar{a} \in \bar{A}'^p$ thus giving $\bar{A}'^{p+0} \subset \bar{A}'^p$. Hence $\bar{A}'^{p+0} = \bar{A}'^p$.

NOTE. This theorem is due to Deheuvels [1b].

LEMMA. *The filtrations defined by $\bar{A}^p$ and $\bar{A}'^p$ on A/I are the same and equal to $\bar{f}$.*

PROOF. Let the filtration defined by $\bar{A}'^p$ be denoted by f'. Then

$$\begin{aligned} f'(\bar{a}) &= \inf\{p \mid a \in \bar{A}'^p\} \\ &= \inf\{p \mid a \in A'^p, a \in \bar{a}\} \\ &= \inf_{a \in \bar{a}} \{\inf(p \mid a \in A'^p)\} \\ &= \inf_{a \in \bar{a}} (f(a)) = \bar{f}(\bar{a}). \end{aligned}$$

Similarly the other one is equal to $\bar{f}$.

3.6. We now turn to consider sufficient conditions for the sequences defined in 2.3 and 3.4 to be exact. First we give a simple criterion using Theorem 2.3.5.

THEOREM 3.6.1. *Sequences* (i) – (v) [see Theorem 1.2.2] *defined for G_2 of E's are exact if for all p and each q, there exists a q' such that for q'' satisfying $q' < q'' < q$, we have $E(p, q'' + 0, q'', q'') = 0$.*

We reduce this to the condition of Theorem 2.3.5 using the following lemma, and the induced filtration of $E(p, q, r, s)$ with which is associated the decomposition $\operatorname{im}\{E(p, q', r, s) \to E(p, q, r, s)\}$, $q \geqslant q'$ which is defined in 3.3.

LEMMA. *If $\{A^p\}$ is a decomposition associated with a filtration f, and if for each q there exists a q' such that for $q' > q'' > q$, $A^{q''+0} = A^{q''}$, then $A^{q''} = A^{q'''}$ for $q' > q''' > q'' > q$.*

PROOF. For if $A^{q''} \neq A^{q'''}$ there exists $a \in A^{q''}$ such that $a \notin A^{q'''}$. Let $f(a) = q^{\text{iv}}$, then $q < q^{\text{iv}} < q'$. Hence $A^{q^{\text{iv}}+0} \neq A^{q^{\text{iv}}}$ which is contrary to the assumption.

PROOF OF THEOREM 3.6.1. Considering

$$0 \to E(p, q'', r, s) \to E(p, q'' + 0, r, s) \to E(p, q'' + 0, q'', s),$$

and

$$0 \to E(p, q'' + 0, q'', s) \to E(p, q'' + 0, q'', q''),$$

it follows that if

$$E(p, q'' + 0, q'', q'') = 0, \text{ then } E(p, q'' + 0, q'', s) = 0,$$

and consequently

$$E(p, q'', r, s) \approx E(p, q''', r, s).$$

Hence $E(p, q'', r, s) \approx E(p, q''', r, s)$, for $q < q'' < q''' < q'$, which proves the theorem.

To consider another sufficient condition we make some more definitions. Let, as usual, $\{A^p\}$ be a decomposition of $\{A, d\}$. We define a new decomposition with R' as indexing set as follows (see 2.1) if $\omega = p$, $A^\omega = A^p$ and if $\omega = p + 0$, $A^\omega = A^{p+0}$. It is easily seen that $\{A^\omega\}$ is a decomposition, and therefore according to the procedure of 3.1 it follows that $H(\omega_1, \omega_2)$ is an exact homology theory. We change the notation a little and write $p + 0$ as $\bar{p}$ when we use it in this context. $E(p, q, r\ s)$ can then be decomposed by

$$\text{im}\{E(p, \omega, r, s) \to E(p, \omega_1, r, s\} = E(p, \omega, r, s) = \{\bar{A}^\omega\}$$

for $\omega \leqslant \omega_1$. To the decomposition $\{\bar{A}^\omega)$ is associated a completed group $\{\bar{A}^{\omega+0}\}$ (Definition 3.2.4) given by

$$\bar{A}^{\omega+0} = \bigcap_{\omega' > \omega} E(p, \omega', r, s) = \operatorname*{inv\,lim}_{\omega' > \omega} E(p, \omega', r, s).$$

But since R is cofinal in R' we have [2, p. 219], when $\omega = \bar{q}'$,

$$\operatorname*{inv\,lim}_{\omega' > \bar{q}'} E(p, \omega', r, s) = \operatorname*{inv\,lim}_{q'' > q'} E(p, q'', r, s) = E(p, q' + 0, r, s).$$

Definition 3.6.2. *A filtration of (A, d) is called accessible if* $\bar{A}^{\bar{q}'+0} = \bar{A}^{\bar{q}'}$, i.e. $E(p, \bar{q}', r, s) = E(p, q' + 0, r, s)$.

We shall paraphrase this in terms involving C's and D's, and obtain certain consequences.

$$E(p, q', r, s) = \text{im}\ \{H(q', s) \to H(p, r)\} = \frac{C_s^{q'} + dA^p + A^r}{dA^p + A^r},$$

$$E(p, \bar{q}', r, s) = \text{im}\ \{H(\bar{q}', s) \to H(p, r)\} = \frac{C_s^{q'+0} + dA^p + A^r}{dA^p + A^r}.$$

Now f and the corresponding decomposition $\{A^p\}$ define a filtration and a decomposition in $C^q_s + dA^p + A^r$, the latter being given by $C^{q'}_s + dA^p + A^r$ and the components of the corresponding completed group being given by $C^{q'+0}_s + dA^p + A^r$. Therefore $E(p, q', r, s)$ and $E(p, \bar{q}', r, s)$ are corresponding decompositions of $\dfrac{C^q_s + dA^p + A^r}{dA^p + A^r}$

The above definition implies that the latter should be attained. But by Theorem 3.5.3 this means that for any $a \in C^q_s + dA^p + A^r$.

$$f(\bar{a}) = f(h),\ h \in a + dA^p + A^r.$$

Therefore we have the following

LEMMA 3.6.4. *A filtration is accessible if*

$$\inf_{a' \in a + dA^p + A^r} f(a') = \inf_{h \in a + dA^p + A^r} f(h).$$

LEMMA. *If the filtration is accessible, then the decompositions of $D(p, q, r, s)$ defined by*

$$\operatorname{im}\{D(\bar{p}', q, r, s) \to D(p, q, r, s)\} = D(\bar{p}', q, r, s),\ \bar{p}' < p$$
$$\operatorname{im}\{D(p, q, \bar{r}', s) \to D(p, q, r, s) = D(p, q, \bar{r}', s),\ \bar{r}' < r,$$

are attained.

PROOF.

$$D(\bar{p}', q, r, s) = \operatorname{im}\{H(\bar{p}', r) \xrightarrow{d} H(q, s)\} = \frac{C^q_r \cap dC^{p'+0}_s + dA^p + A^s}{dA^q + A^s}$$

This is attained if

$$\inf_{a' \in a + dAq^s + A} f(a') = \inf_{h \in a + dAq + A^s} f(h),$$

which is true in virtue of Lemma 3.6.4. Similarly it follows that the second is attained.

COROLLARY. $D(p + 0, q, r, s) = D(\bar{p}, q, r, s)$
$D(p, q, r + 0, s) = D(p, q, \bar{r}, s)$.

A number of sequences of Theorems 3.4.2 become exact. They can be obtained by considering diagrams of the following kind:

$$\begin{array}{ccccccccc} 0 \to & E(p, q, r, s) & \to & E(p, q + 0, r, s) & \to & E(p, q + 0, q, s) & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & E(p, q, r, s) & \to & E(p, \bar{q}, r, s) & \to & E(p, \bar{q}, q, s) & \to 0. \end{array}$$

Since the vertical maps are isomorphisms, the above sequence is also exact. But we cannot say that all the sequences will be exact. At best, sequences involving two terms in R' can be proved to be so. We shall restrict ourselves to the important case of the exact sequences enunciated in Theorem 3.4.2.

THEOREM 3.6.5. *Let*

$$E(p+h+0, p+0, p, p-h) = L_p(h), \quad L(h) = \Sigma L_p(h);$$
$$E(p+h+0, p+0, p, -\infty) = P_p(h), \quad P(h) = \Sigma P_p(h);$$
$$E(\infty, p+0, p, -\infty) = K_p, \quad K = \Sigma K_p;$$
$$D(p+0, p, p-h, -\infty) = M_p(h), \quad M(h) = \Sigma M_p(h);$$
$$D(\infty, p+h+0, p+0, p) = N_p(h), \quad N(h) = \Sigma N_p(h).$$

If the filtration is accessible and R has a countable, cofinal subset, then the following sequences are exact

(i) $0 \to P \to L \to M \to 0,$

(ii) $0 \to N \to P \to K \to 0,$

(iii) $0 \to E(\infty, p, -\infty, -\infty) \to E(\infty, p+0, -\infty, -\infty) \to E(\infty, p+0, p, -\infty) \to 0,$

(iv) $0 \to D(p+0, p, q, -\infty) \to D(p+0, p, q+0, -\infty) \to D(p+0, p, q+0, q) \to 0,$

(v) $0 \to D(q, p+0, p+0, p) \to D(q+0, p+0, p+0, p) \to D(q+0, q, p+0, p) \to 0.$

PROOF. By Lemma 3.6.4. and its corollary,

$$M_p = D(p+0, p, p-h, -\infty) = D(\bar{p}, p\, p-h, -\infty),$$
$$K_p = E(\infty, p+0, p, -\infty) = E(\infty, \bar{p}, p, -\infty),$$
$$L_p = E(p+h+0, p+0, p, p) = \operatorname*{inv\,lim}_{p'>p+h} E(p', p+0, p, p)$$
$$= \operatorname*{inv\,lim}_{p'>p+h} E(p', \bar{p}, p, p),$$
$$D(p+0, p, q+0, q) = D(\bar{p}, p, \bar{q}, q).$$

Now we know that

$$0 \to E(p', \bar{p}, p, p) \to E(p', \bar{p}, p, -\infty) \to D(\bar{p}, p, p, -\infty) \to 0$$

is exact. Hence by Theorem 2.3.3

$$0 \to \operatorname*{inv\,lim}_{p'>p+h} E(p',\bar{p},p,p) \to \operatorname*{inv\,lim}_{p'>p} E(p',\bar{p},p,-\infty) \to D(\bar{p},p,p,-\infty)$$

is exact. But

$$E(p',\bar{p},p,-\infty) \to E(p'',\bar{p},p,-\infty)$$

is onto, and R has a countable cofinal subset. Hence [2, p. 218]

$$\operatorname*{inv\,lim}_{p'>p+h} E(p',\bar{p},p,-\infty) \to E(p',\bar{p},p\ -\infty)$$

is onto. But then from the commutative diagram

$$\begin{array}{ccc} \operatorname*{inv\,lim}_{p'>p+h} E(p',\bar{p},p,-\infty) & & \\ \downarrow \text{onto} & \searrow & \\ E(p',\bar{p},p,-\infty) & \xrightarrow[\text{onto}]{} & D(p,\bar{p},p,-\infty), \end{array}$$

it follows that the third map is also onto. Hence (i) is proved

(ii) also follows as above, while (iii), (iv), (v) are immediate as $H(\omega_1, \omega_2)$ defined above is an exact homology theory.

COROLLARY. *Under the same conditions as in Theorem* 3.5.3 *we have the Morse relations.*

III. Applications to topological spaces.

4. We now apply the previous considerations in a variety of ways to topological spaces. Let X be a topological space and f be a real-valued function on it. Let $X^p = \{x \mid f(x) \leqslant p\}$ Let further a homology theory be given on X which satisfies the Eilenberg-Steenrod axioms [2, p 11]. In fact at the present level only algebraic axioms are required. The other axioms will be required (i) to link the critical groups at different levels, which we shall define in the sequel, to the critical groups at critical points—more precisely, we shall require excision axioms, and (ii) to link the critical groups with the groups $H(f \leqslant e, f < c)$, which mostly will be done with the help of the homotopy axiom. But we shall not occupy ourselves in this work with these problems.

4.1. Let $H(p, q) = H(X^p, X^q)$. If $(p, q) \geqslant (p', q')$, let $\pi^{p, q}_{p', q'}$ be the homomorphism $H(X^{p'}, X^{q'}) \to H(X^p, X^q)$, and for each (p, q, r) let d_{pqr} be the homomorphism $H(X^p, X^q) \xrightarrow{d} H(X^q, X^r)$

THEOREM 4.1.1. *$H(p, q)$, $\pi^{p, q}_{p', q'}$, d_{pqr} form a homology theory with R as indexing set.*

Proof follows from axioms 1, 2, 3, 4 Eilenberg-Steenrod [2 p. 11].

If we now consider the extension by direct limits, we have the exact sequences of Theorem 1.2.2 and, in particular, those given by Theorem 3 4.2 In order to have the Morse relations we make the following

DEFINITION 4.1.2. *A level $p \in R$ is a direct homological critical level if $E(p, p, p-0, p-0) \neq 0$ and $E(p, p, p-0, p-0)$ is a direct homological critical group at level p.*

The group $E(p, p, p-0, p-0) = \operatorname{dir} \lim\limits_{p' < p} \{ H(X^{p'}) \to H(X^p) \}$ is a cap group at level p defined by Morse [6b]. We have obtained his relation in the most general situation without any assumptions on homology (besides exactness) or compactness of X^p

4.2. Before passing to consider the inverse extensions we wish to consider the case of cohomology groups. Since the maps go in the opposite way to those of homology groups, we need a little different theory than the one given above, in which we use maps going in a direction opposite to that of the order relation in R. We proceed as follows

DEFINITION 4.2.1. *We define couples, triples (p, q), (p, q, r) if $p \leqslant q$, $p \leqslant q \leqslant r$ and so on The corresponding homomorphisms are $\pi^{p, q}_{p', q'} : H(p', q') \to H(p, q)$, if $(p, q) \leqslant (p', q')$, and $d_{pqr} : H(p, q) \to H(q, r)$*

Now the terms $H(p, q)$, $\pi^{p, q}_{p', q'}$, d_{pqr} are the same as those in Definition 1.1.1. Hence if the four axioms are satisfied we have a homology theory with $(R, \leqslant)$ as indexing set, and therefore all the exact sequences in Theorems 1.2.2, 3.4.2 hold.

Let $\mathcal{H}$ denote the cohomology groups. Let

$$H(p,q) = \mathcal{H}(X^q, X^p),\ p \leqslant q;$$

$$\pi^{p,\,q}_{p',\,q'} : H(p',q') \to H(p,q), \text{ for } (p,q) \leqslant (p',q').$$

be given by $\mathcal{H}(X^{q'}, X^{p'}) \to \mathcal{H}(X^q, X^p)$ and $d_{pqr} : H(p,q) \to H(q,r)$ be given by $\mathcal{H}(X^q, X^p) \xrightarrow{d} \mathcal{H}(X^r, X^q)$.

THEOREM 4.2.2. $\{H(p,q), \pi^{p,\,q}_{p',\,q'}, d_{pqr}\}$ *is a homology theory with* R *as indexing set.*

The groups in the direct extension become $E(p+0, q+0, r+0, s+0)$ and those in the inverse extension $E(p-0, q-0, r-0, s-0)$.

We require corresponding alterations in the definition of decomposition and filtration. For decomposition we require $A^p \supset A^q$ if $p < q$ and $dA^p \subset A^p$; for filtration $f(a-b) = \min(f(a), f(b))$ and $f(da) \geqslant f(a)$. The decomposition of $E(p,q,r,s)$ is given by im $E(p,q',r,s) \to E(p,q,r,s)$, $q' > q$. A level p is said to be *a direct cohomology critical level* if $E(p+0,\ p+0,\ p,\ p) \neq 0$ and $E(p+0,\ p+0,\ p,\ p)$ is *the direct cohomological critical group* at level p.

4.3. Now if we consider the inverse extension of homology theories, we see that since the extended exact sequences do not hold in general, Morse's relations cannot be generalized as they stand. Since we know some sufficient conditions for these to hold in particular cases considered in 3.6, we proceed to consider filtrations of chain and cochain groups of X. We do this for the case of singular cohomology, and prove that a filtration to be defined is accessible, and obtain the exact sequences as in Theorem 1.2 2 and 3 4.2, and thus generalize Morse's relations for this case.

Let S be the group of singular chains, and $\mathcal{S}$ be the group of singular cochains. Let S be decomposed by $S^p = S(X^p)$ and $\mathcal{S}$ by $\mathcal{S}^p = \mathcal{S}(X, X^p)$, where $\mathcal{S}(X^p)$ is the group of singular chains whose support lies in X^p and $\mathcal{S}(X, X^p)$ is the group of singular cochains which vanish on the singular simplices whose support lies in X^p.

REMARK. We can define a homology theory corresponding to the decomposition $\{S^p\}$ of S and $\{\mathscr{L}^p\}$ of $\mathscr{L}$ as in 3.1, and this will be the same as that considered in 4.1, and 4.2 for $H(X^p, X^q) = H(S^p/S^q)$ in the case of homology, and for the cohomology

$$H(X^q, X^p) = \mathscr{H}\frac{(\mathscr{L}/\mathscr{L}(X^q))}{(\mathscr{L}/\mathscr{L}(X^p))} = \mathscr{H}(\mathscr{L}(x^p)/\mathscr{L}(X^q)).$$

The filtrations corresponding to these decompositions are: for singular chains,

$$f(s) = \inf\{p \mid |s| \subset X^p\},$$

and, for singular cochains,

$$\begin{aligned} f(\xi) &= \sup\{p \mid \xi(s) = 0 \text{ for all } s \text{ such that } |s| \subset x^p\} \\ &= \sup\{p \mid (\xi/X^p) = 0\}, \end{aligned}$$

where ξ/X^p denotes the restriction of ξ to X^p.

REMARK. Though f defined above is the filtration associated with the decompositions $\{S^p\}$, $\{\mathscr{L}^p\}$; the decompositions associated with this filtration do not coincide with them. In fact we have

THEOREM 4.3.2. *$\{S^p\}$ is the completion of the decomposition of S obtained from f as defined above, if f is upper-semi-continuous.*

PROOF. (a) For singular chains: The decomposition $\{S'^p\}$ corresponding to the filtration f is, by definition, given by $s \in S'^p$ if $f(s) < p$, and its completion by $s \in S'^{p+0}$ if $f(s) \leqslant p$, i.e. maximum of f on the support of s is $\leqslant p$. Hence $s \in S^p$. Now the support of a singular chain is compact, and therefore for an upper-semi-continuous function the maximum is attained there. Hence $f(s) = p$. So $s \in S^p$ if and only if $f(s) \leqslant p$, i.e. $S^p = S'^{p+0}$.

(b) For singular *cochains*: the filtration of singular cochains can be paraphrased as follows: $f(\xi) = p$ *if* ξ is zero on all singular simplices on which maximum of f is less than p, and there is at least one singular simplex on which l.u.b of f is p. Now if f is upper semi-continuous, p is attained on that singular simplex.

$\mathscr{L}'^p = \{\xi \mid \xi$ vanishes on all simplices on which f is less than or equal to p but does not vanish at least on one singular simplex in $X^{p'}$, $p' > p\} = \bigcup_{p'>p} \mathscr{L}(x^{p'})$. Hence $\mathscr{L}'^{p-0} = \mathscr{L}(X^p)$.

THEOREM 4.3.3. *The filtration defined above for singular cochains is accessible.*

PROOF. We require to prove that the decomposition of $E(p, q, r, s)$ defined by $\text{im}(E(p, \bar{q}', r, s))$ (for notation see 3.6) is attained. But this decomposition is that defined on $\dfrac{C^q{}_s + d\,A^p + A^r}{d\,A^p + A^r}$ starting from the decomposition of $C^q{}_s$ defined by $C^{\bar{q}'}{}_s$ for $\bar{q}' > q$

By Theorem 3.5.3 we need to prove that for any $\xi \in C^q_s$,

$$\bar{f}(\bar{\xi}) = \sup_{\xi' \in \xi + dA^p + A^r} \{f(\xi')\} = \underset{\eta \in \xi + dA^p + A^r}{f(\eta)}$$

Let $\bar{f}(\bar{\xi}) = h$, $f(\xi) = h'$. By the above definition there exists $\xi_n \in d\,A^p + A^r$ such that $f(\xi + \xi_n) = h_n$, where $h_n < h$ is a sequence tending to h. Now, since $f(\xi) = h'$, ξ vanishes on all simplices where l.u.b. of f is less than h' and does not vanish at least on one singular simplex with l.u.b of f equal to h'.

Since $f(\xi + \xi_n) = h_n$, the same is true for singular simplices with l.u.b $< h_n$. We denote by C'_n the set of singular simplices with l.u.b. of $f < h_n$ and by $C^m{}_n$ the set of singular simplices with l.u.b between h_m and h_n. We see that $\xi + \xi_n$ is zero on singular simplices, of C'_n, i.e. on these singular simplices ξ_n has values equal to that of ξ with opposite sign.

Define a new cochain ξ'_n which on singular simplices of C'_n is $-\xi$ and is zero elsewhere.

(i) ξ'_n coincides with ξ'_m for $m \leqslant n$ on the singular simplices of C'_m.

Let ξ' be a new cochain which on the singular simplices of C'_n is ξ'_n given above, and zero elsewhere. The definition is valid because

of (i). $f(\xi + \xi') \geqslant h$ for it vanishes on any singular simplex in $X^{h'}$, $h' < h$. If we prove $\xi' \in dA^p + A^r$, then it would follow that $f(\xi + \xi') \leqslant h$, hence $f(\xi + \xi') = h$ proving our result. To prove this, let $\xi_n = da^p{}_n + a_n{}^r$, where $a^p{}_n \in A^p$, $a_n{}^r \in A^r$. Let a'^p_n, a'^r_n be singular cochains which on C'_n take the same values as those of $a^p{}_n$, $a'_n{}^r$ and zero elsewhere. Let $\xi'_n = da'_n{}^p + a'^r_n$. Let $a'^p{}_{n,n-1}$ be the cochain which takes on $C'_n{}^{n-1}$ values which are the same as those of $a'^p{}_n$ and zero elsewhere. Similarly let $a'^r{}_{n,n-1}$ be defined. Then let a'^p be a singular cochain which takes on $C'_n{}^{n-1}$ values which are the same as those of $a'^p{}_{n,n-1}$ and zero elsewhere, and similarly for a'^r. Then $\xi' = da'^p + a'^r$, which proves that $\xi' \in dA^p + A^r$.

4.4. In the preceding section we considered two generalizations of the classical definition of critical groups, namely $H(f \leqslant c, f < c)$. We prove that the two become identical in the following situation :

THEOREM 4.4.1. *Let for each c there exist an $\epsilon > 0$ such that $H(f < c, f \leqslant c - \epsilon) = 0$ (or $H(f \leqslant c + \epsilon, f < c) = 0$). Then the direct cap groups at level $c = H(f \leqslant c, f < c)$ (or the Inverse cap groups $= H(f \leqslant c, f < c)$). Further, if there exists $\epsilon > 0$ for which both conditions hold at the same time, then direct and inverse cap groups are isomorphic.*

PROOF. From the exact sequence $0 \to H(f < c, f \leqslant c - \epsilon) \to H(f \leqslant c, f \leqslant c - \epsilon) \to H(f \leqslant c, f < c) \to H(f < c, f \leqslant c - \epsilon) \to 0$ it follows that $H(f \leqslant c, f \leqslant c - \epsilon) \approx H(f \leqslant c, f < c)$ Hence $H(c, c - \epsilon) \approx H(c, c - \epsilon')$, so that

$$\operatorname*{dir\,lim}_{\epsilon > 0} H(c, c - \epsilon) \approx H(c, c - \epsilon) \approx H(f \leqslant c, f < c).$$

Similarly the second assertion follows. Combining the two we have the third.

Comparing this with the situation (1), (2) in the Introduction, we see that we get the classical results from Theorem 3.4.2.

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